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MELLIN CONVOLUTION FOR SIGNAL FILTERING
AND ITS APPLICATION TO THE GAUSSIANIZATION OF LÉVY NOISE

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ABSTRACT

Noises are usually assumed to be Gaussian so that many existing signal processing techniques can be applied with no worry. However, in many real world natural or man-made systems, noises are usually heavy-tailed. It is increasingly desirable to address the problem of finding an opportune filter function for a given input noise in order to generate a desired output noise. By filtering theory, the probability density function of the output noise can be expressed by the integral of the product of the density of the input noise and the filter function. Adopting Mellin transformation rules, the Mellin transform of the unknown filter is determined by the Mellin transforms of the known density of the input noise and the desired density for the output noise. Finally, after the inversion, the Mellin–Barnes integral representation of the filter function is derived. The method is applied to compute the filter function to convert a Lévy noise into a Gaussian noise.

INTRODUCTION

In many real world natural or man-made systems, noises are usually heavy-tailed [1] with lots of spikes and for this they are difficult to be managed and should be further processed [2]. For example, the networked induced delays in networked control systems (NCS) are of spiky nature which hints that the generating dynamics should be characterized by fractional order differential models as pointed out in [3] based on [4]. Therefore, in fractional order control [5] and fractional order signal processing [1], it is increasingly desirable to address the problem of finding an opportune filter function for an input noise of a given distribution in order to generate an output noise of the desired distribution. In this contribution, noises with continuos probability density function (PDF) are taken into account. The Mellin transform tool is considered and in particular its convolution-type integral, where as convolution integral is intended that integral including the product of two functions whose transform is the product of the corresponding transformed functions. By filtering theory, the PDF of the output noise can be expressed by the integral of the product of the density of the input noise and the filter function. Adopting Mellin transformation rules, the Mellin transform of the unknown filter can be determined by the Mellin transforms of the known density of the input noise and the desired density for the output noise. Finally, after the inversion, the Mellin–Barnes integral representation of the filter function is derived. The method is applied to compute the filter function to convert a Lévy noise into a Gaussian noise.

MELLIN CONVOLUTION FOR SIGNAL FILTERING

Mellin transform is a powerful mathematical tool that better than other methods permits to successfully evaluate integrals [6–

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8], derive subordination formula [9, 10], naturally related to the Mellin–Barnes integral representation of Special Functions [6, 11, 12] and then also to manage higher transcendental functions and the well-known H-function [13–15], see Appendix.

The Mellin transform of a function $\psi(x)$, for $x > 0$, is defined as

$$\psi^*(s) = \int_0^\infty \psi(x) x^{s-1} dx, \quad s \in C, \tag{1}$$

and the antitransformation formula reads

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \psi^*(s) x^{-s} ds, \quad \sigma = \mathscr{Re}(s). \tag{2}$$

The transformed function $\psi^*(s)$ exists if the integral $\int_0^\infty |\psi(x)| x^{s-1} dx$ is bounded and this constraint is met in the vertical strip $a < \sigma = \mathscr{Re}(s) < b$, where the boundary values a and b follow from the analytic structure of $\psi(x)$ provided that $|\psi(x)| \leq Mx^{-a}$ when $x \to 0^+$ and $|\psi(x)| \leq Mx^{-b}$ when $x \to +\infty$. Hereinafter, the Mellin transformation pair is denoted by

$$\psi(x) \xleftrightarrow{\mathscr{M}} \psi^*(s). \tag{3}$$

Please see specialized treatises and/or handbooks [6,11] for more details.

Applying residue theorem to (2), the original function $\psi(x)$ has the following series representation

$$\psi(x) = \sum_{k=1}^n \operatorname*{Res}_{s=\xi_k} \{\psi^*(s)\} x^{-\xi_k}, \tag{4}$$

where Res stands for residue and ξ_k, with $k = 1, \ldots, n$, are the n poles of the transformed function $\psi^*(s)$. This means that antitransformation formula (2) can be re-written as

$$\psi(x) = \frac{1}{2\pi i} \int_{\mathscr{L}} \psi^*(s) x^{-s} ds, \tag{5}$$

where $\mathscr{L}$ is the integration path that encircles all the poles ξ_k of the integrand $\psi^*(s)$.

Here, it is called *convolution* that operator involving two functions whose trasformation is given by the product of the transformation of two involved functions. What concerns Mellin transform the convolution is

$$\psi(x) = \int_0^\infty \psi_1\left(\frac{x}{\eta}\right) \psi_2(\eta) \frac{d\eta}{\eta} \xleftrightarrow{\mathscr{M}} \psi_1^*(s) \psi_2^*(s). \tag{6}$$

Let $p_i(x,t)$ and $p_o(x,t)$ be the PDF of the input and output noise, respectively, where $x \in R$ is the random fluctuation of the noisy signal and $t \in R^+$ is an associated positive parameter as for example the elapsed time. Then a general input noise p_i can be transformed into p_o by an opportune filtering function f through the subordination type formula

$$p_o(x,t) = \int_0^\infty p_i(x,\tau) f(\tau,t) d\tau. \tag{7}$$

If the statistical self-similarity for both input and output signals is assumed, then, for the opportune choice of a scaling factor depending on t, the statistical description of the signals emerges to be scale invariant. Then, since the whole statistical information is included inside the PDF, it means that the PDFs of the input and output signals can be expressed in the following scale invariant arrangement

$$p_i(x,\tau) = \tau^{-\gamma_i} \varphi_i\left(\frac{x}{\tau^{\gamma_i}}\right), \tag{8}$$

$$p_o(x,t) = t^{-\gamma_o} \varphi_o\left(\frac{x}{t^{\gamma_o}}\right), \tag{9}$$

where τ^{γ_i} and $\varphi_i(x/\tau^{\gamma_i})$ embodied the scale factor and the scale invariant functional form of $p_i(x,\tau)$, τ^{γ_i} and $\varphi_o(x/\tau^{\gamma_o})$ have the same meaning for $p_o(x,t)$. Finally, formula (7) becomes

$$t^{-\gamma_o} \varphi_o\left(\frac{x}{t^{\gamma_o}}\right) = \int_0^\infty \tau^{-\gamma_i} \varphi_i\left(\frac{x}{\tau^{\gamma_i}}\right) f(\tau,t) d\tau. \tag{10}$$

Formula (10) can be understood as a convolution integral (6) after straightforward change of variable.

Consider the positive semi-axes $x > 0$. Applying Mellin transform (1) to both sides of (10) gives

$$t^{-\gamma_o} \int_0^\infty \varphi_o\left(\frac{x}{t^{\gamma_o}}\right) x^{s-1} dx =$$
$$\int_0^\infty \tau^{-\gamma_i} \left\{ \int_0^\infty \varphi_i\left(\frac{x}{\tau^{\gamma_i}}\right) x^{s-1} dx \right\} f(\tau,t) d\tau, \tag{11}$$

from which it follows that

$$\int_0^\infty f(\tau,t) \tau^{\gamma_i(s-1)} d\tau = t^{\gamma_o(s-1)} \frac{\varphi_o^*(s)}{\varphi_i^*(s)}, \tag{12}$$

where $\varphi_i^*(s)$ and $\varphi_o^*(s)$ are the Mellin transform of φ_i and φ_o, respectively. Antitransformation of (12) by using (5) gives

$$f(\tau,t) = \frac{\gamma_i}{\tau} \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\varphi_o^*(s)}{\varphi_i^*(s)} \left(\frac{\tau^{\gamma_i}}{t^{\gamma_o}}\right)^{-s+1} ds, \tag{13}$$

2

Formula (13) can be analogously written as

$$f(\tau,t) = \frac{\gamma_i}{\tau} \frac{\tau^{\gamma_i}}{t^{\gamma_o}} \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\varphi_o^*(s)}{\varphi_i^*(s)} \left(\frac{\tau^{\gamma_i}}{t^{\gamma_o}}\right)^{-s} ds \qquad (14)$$

$$= \frac{\gamma_i}{\tau} \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\varphi_o^*(s+1)}{\varphi_i^*(s+1)} \left(\frac{\tau^{\gamma_i}}{t^{\gamma_o}}\right)^{-s} ds, \qquad (15)$$

where $\mathscr{L}$ is the proper integration path for each integral. Formula (13) emerges to be the desired filter function which transforms an input noise distributed as $p_i(x,t)$ into an output noise distributed as $p_o(x,t)$.

GAUSSIANIZATION OF LÉVY NOISE

The formalism discussed in the previous section can be used to calculate the filter that given an input Lévy noise outputs a Gaussian noise.

This means that, with reference to (8), the input noise is

$$\gamma_i = \frac{1}{\alpha}, \quad p_i(x,\tau) = \tau^{-1/\alpha} \varphi_i\left(\frac{x}{\tau^{1/\alpha}}\right), \qquad (16)$$

where $0 < \alpha \le 2$ is the characteristic exponent, and the Mellin transform [9, see formula (5.1)]

$$\varphi_i^*(s) = \frac{1}{\alpha} \frac{\Gamma(1/\alpha - s/\alpha)\Gamma(s)}{\Gamma(1-\rho+\rho s)\Gamma(\rho-\rho s)}, \quad \rho = \frac{\alpha-\theta}{2\alpha}, \qquad (17)$$

where θ is the skewness parameter $|\theta| \le \min\{\alpha, 2-\alpha\}$. In order to highlight these parameters, hereinafter Lévy density will be denoted by $L_\alpha^\theta(x,t)$. It is well-known that the extension to $x < 0$ is given by the exchange $\theta \to -\theta$.

Since it is desired a Gaussian output noise, $p_o(x,t)$ is

$$p_o(x,t) = \mathscr{G}(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, \qquad (18)$$

so that with reference to (9)

$$\gamma_o = \frac{1}{2}, \quad \varphi_o\left(\frac{x}{t^{1/2}}\right) = \frac{1}{2\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left(\frac{x}{t^{1/2}}\right)^2\right\}, \qquad (19)$$

and

$$\varphi_o^*(s) = \frac{1}{2} \frac{\Gamma(s)}{\Gamma(1/2+s/2)}. \qquad (20)$$

Finally, inserting (17) and (20) in (13), the Gaussianization of the Lévy noise is obtained by the filter function

$$f(\tau,t) = \frac{1}{2\tau} \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\Gamma(1+\rho s)\Gamma(-\rho s)}{\Gamma(-s/\alpha)\Gamma(1+s/2)} \left(\frac{\tau^{1/\alpha}}{t^{1/2}}\right)^{-s} ds. \qquad (21)$$

The filter function emerges to be a self-similar function with the following scaling law

$$f(\tau,t) = t^{\alpha/2} \mathscr{F}\left(\frac{\tau}{t^{\alpha/2}}\right). \qquad (22)$$

Formula (21) is the Mellin–Barnes integral representation of the Gaussianing filter function. The integration path $\mathscr{L}$ encircles the poles of $\Gamma(1+\rho s)$. The filter $f(\tau,t)$ can be expressed in terms of H-function, see Appendix, and it turns out to be

$$f(\tau,t) = \frac{1}{2\tau} H_{22}^{11}\left[\frac{\tau^{1/\alpha}}{t^{1/2}} \,\middle|\, \begin{matrix}(1,\rho)(1,1/2)\\(1,\rho)(1,1/\alpha)\end{matrix}\right], \qquad (23)$$

and then the *Gaussianizing formula* for a Lévy noise is

$$\mathscr{G}(x,t) = \int_0^\infty L_\alpha^\theta(x,\tau) f(\tau,t) d\tau, \qquad (24)$$

or analogously

$$\mathscr{G}(x,t) = \frac{1}{2} \int_0^\infty L_\alpha^\theta(x,\tau) H_{22}^{11}\left[\frac{\tau^{1/\alpha}}{t^{1/2}} \,\middle|\, \begin{matrix}(1,\rho)(1,1/2)\\(1,\rho)(1,1/\alpha)\end{matrix}\right] \frac{d\tau}{\tau}. \qquad (25)$$

With reference to the Appendix, the existence condition for (23) requires that

$$\mu = \frac{1}{\alpha} - \frac{1}{2} = \frac{2-\alpha}{2\alpha} > 0,$$

which holds for $\alpha \le 2$, that means for all values of Lévy characteristic exponent α.

In order to express the filter function (21) by a series, applying residue theorem to (21) gives

$$f(\tau,t) = \frac{1}{\tau} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{\Gamma(k)}{\Gamma[k/(\alpha\rho)]\Gamma[-k/(2\rho)]} \left(\frac{\tau^{1/\alpha}}{t^{1/2}}\right)^{k/\rho}. \qquad (26)$$

3

Using the Gamma function property $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$ for the term $\Gamma[-k/(2\rho)]$, formula (15) becomes

$$f(\tau,t) = \frac{-1}{\pi\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k)\Gamma\left(1+\frac{k}{2\rho}\right)}{\Gamma\left(\frac{k}{\alpha\rho}\right)}\sin\left(\pi\frac{k}{2\rho}\right)\left(\frac{\tau^{1/\alpha}}{t^{1/2}}\right)^{k/\rho}.$$

$$(27)$$

It is worth noting to remark that expression (27) is more convenient for numerical computation than (26). In fact, the parameter ρ can be a rational number or be truncated to a rational number by the computer CPU, then there exist such k_* for those $k_*/(2\rho) \in N$ and then $\Gamma[-k_*/(2\rho)] \to \infty$ which creates problem for the numerical computation of series (26). This problem is avoided in (27) because for the same k_* it results $\sin[\pi k_*/(2\rho)] = 0$.

As briefly discussed in the Appendix, the asymptotic expansion of $f(\tau,t)$ for $\tau/t^{\alpha/2} \to \infty$ is obtained by chosing an integration path in (21) that encircles all poles of $\Gamma(-\rho s)$. Then it results

$$f(\tau,t) = \frac{-1}{\pi\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k)\Gamma\left(1+\frac{k}{\alpha\rho}\right)}{\Gamma\left(\frac{k}{2\rho}\right)}\sin\left(\pi\frac{k}{\alpha\rho}\right)\left(\frac{\tau^{1/\alpha}}{t^{1/2}}\right)^{-k/\rho}.$$

$$(28)$$

Special Cases

In the symmetric case $\theta = 0$, such that $\rho = 1/2$, and then applying residue theorem to (21) gives for $\tau/t^{\alpha/2} \to \infty$

$$f(\tau,t) = \frac{-1}{\pi\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(1+\frac{2k}{\alpha}\right)\sin\left(\pi\frac{2k}{\alpha}\right)\left(\frac{\tau}{t^{\alpha/2}}\right)^{-2k/\alpha}. \quad (29)$$

The plots of the filter function $f(\tau,t)$ and the Lévy and Gaussian PDFs for this symmetric case with $\theta = 0$ are shown in FIG. 1 for $\alpha = 0.8$ and $t = 1$.

A Lévy PDF with extremal value of the skewness parameter is called *extremal*. It is possible to prove that for $0 < \alpha < 1$ the PDFs are one-sided on the positive semi-axes if $\theta = -\alpha$ and the negative semi-axes if $\theta = \alpha$. Since formula (21) is valid for $x > 0$ here it is considered the extremal PDF for $\theta = -\alpha$ and then $\rho = 1$. In this case the filter function is

$$f(\tau,t) = \frac{-1}{\pi\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k)\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(\frac{k}{\alpha}\right)}\sin\left(\pi\frac{k}{2}\right)\left(\frac{\tau}{t^{\alpha/2}}\right)^{k/\alpha}, \quad (30)$$

and its asymptotic expansion

$$f(\tau,t) = \frac{-1}{\pi\tau}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k)\Gamma\left(1+\frac{k}{\alpha}\right)}{\Gamma\left(\frac{k}{2}\right)}\sin\left(\pi\frac{k}{2}\right)\left(\frac{\tau}{t^{\alpha/2}}\right)^{-k/\alpha}. \quad (31)$$

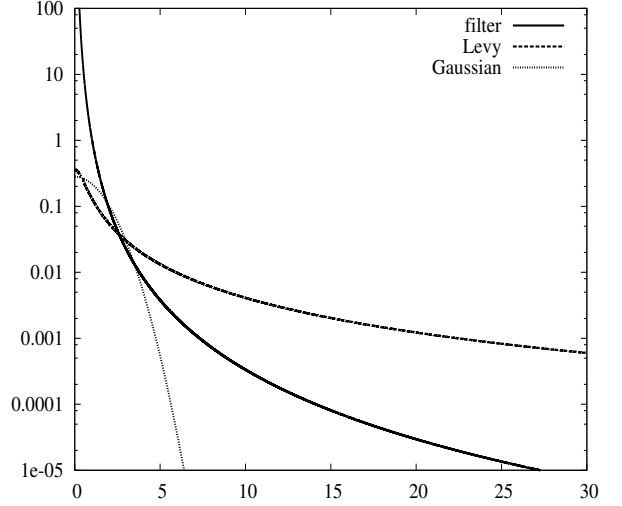**FIGURE 1**. PLOTS OF THE REDUCED FUNCTIONS OF THE FILTER $\mathscr{F}(\xi_f)$ (22), OF THE LÉVY PDF $\varphi_i(\xi_L)$ (16) AND OF THE GAUSSIAN PDF $\varphi_o(\xi_G)$ (19), WITH $\alpha = 0.8$ AND $\theta = 0$, WHERE THE ABSCISSA VALUES ARE $\xi_f = \tau/t^{\alpha/2}$, $\xi_L = x/\tau^{1/\alpha}$ AND $\xi_G = x/t^{1/2}$.

To conclude, when the input noise is Gaussian $\alpha = 2$ and $\theta = 0$, then (21) reduces to

$$f(\tau,t) = \frac{1}{\tau}\frac{1}{2\pi i}\int_{\mathscr{L}}\left(\frac{\tau}{t}\right)^{-s}ds = \delta(\tau - t). \quad (32)$$

In fact, the Mellin transform of $\delta(\tau - t)$ is

$$\int_0^{\infty}\delta(\tau - t)\tau^{s-1}d\tau = t^{s-1}, \quad (33)$$

from which the inversion formula turns out to be

$$\delta(\tau - t) = \frac{1}{2\pi i}\int_{\mathscr{L}}t^{s-1}\tau^{-s}ds$$
$$= \frac{1}{\tau}\frac{1}{2\pi i}\int_{\mathscr{L}}\left(\frac{\tau}{t}\right)^{-s}ds. \quad (34)$$

CONCLUSION

The Mellin convolution integral is used to generally derive the filter function that convert an input noise which is difficult to be managed into a desirable output noise. This method is applied to modify a Lévy noise into a Gaussian noise. The founded filter function is emerged to be an higher trascendental hypergeometric function and its representation as Mellin–Barnes integral is

4

given, as well as the representation in terms of H-function. The asymptotic expansion is computed.

The special cases for symmetric and extremal Lévy densities are also shown with their asymptotic expansions.

The construction of a method which, analogously to that here presented, can be used to convert a general noise into a desired noise, when they follow discrete PDFs, will be the subject of the future development of the research.

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Appendix: H-function

The H-function is defined by means of a Mellin–Barnes type integral as follows [13–15]

$$H_{p,q}^{m,n}\left[z\left|\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{array}\right.\right]=\frac{1}{2\pi i}\int_{\mathscr{L}}h(s)z^{-s}\,ds,\quad(35)$$

with

$$h(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{i=n+1}^{p}\Gamma(a_i+A_is)},\quad(36)$$

where an empty product is always interpreted as unity, $\{m,n,p,q\}\in N_0$ with $1\leq m\leq q$ and $0\leq n\leq p$, $\{A_i,B_j\}\in R^+$ and $\{a_i,b_j\}\in R$, or C, with $i=1,\ldots,p$ and $j=1,\ldots q$ such that $A_i(b_j+k)\neq B_j(a_i-\ell-1)$ with k and $\ell\in N_0$, $i=1,\ldots,n$ and $j=1,\ldots,m$. The poles of the integrand in (35) are assumed to be simple. The integration path $\mathscr{L}$ encircles all the poles of $\Gamma(b_j+B_js)$ with $j=1,\ldots,m$.

The H-function is an analytic function of z and exists for all $z\neq0$ when $q\geq1$ and $\mu>0$ or for $0<|z|<\Delta$ when $q\geq1$ and $\mu=0$ where

$$\mu=\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i,\quad\Delta=\left\{\prod_{i=1}^{p}A_i^{-A_i}\right\}\left\{\prod_{j=1}^{q}B_j^{B_j}\right\}.$$

For other existence conditions see [13–15].

The asymptotic expansion for $|z|\to\infty$ is obtained by integration around the poles of $\Gamma(1-a_i-A_is)$ with $i=1,\ldots,n$. Actually this is similar to exchange $s\to-s$ and then passing from the series of powers of z to a series of powers of $1/z$, from which the asymptotic expansion.

In the particular case with $n = 0$ the asymptotic behaviour is of exponential type and determined for real z by the formula

$$H_{p,q}^{m,0}(z) \simeq \mathscr{O}\left(z^{[\mathrm{Re}(\omega)+1/2]/\mu}\right) \exp\left\{\mu \cos\left(\frac{\zeta \pi}{\mu}\right)\left(\frac{z}{\Delta}\right)^{1/\mu}\right\}, \quad (37)$$

for $z \to \infty$ where

$$\omega = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}, \quad \zeta = \sum_{j=1}^{m} B_j - \sum_{i=n+1}^{p} A_j.$$