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Evolution equation for flame ball radius

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Abstract: A method to derive the evolution equation of the radius of an isolated flame ball is proposed. This new method strongly simplifies and generalizes previous methods which are based on matching of multiple asymptotic expansions. The main idea is to split the flame ball in two components: the inner kernel, which is driven by a Poisson-type equation with a general polynomial forcing term, and the outer part, which is driven by a generalized anomalous diffusion process. The evolution equation for the radius of the flame ball is determined as the evolution equation for the interface that matches the solution of the inner spherical kernel and the solution of the outer diffusive part. The resulting equation emerges to be a nonlinear fractional differential equation which reduces to literature equations when a Gaussian diffusion process and the opportune forcing are considered.

Keywords: Temperature profiles, propagation, integral equation formulation, nonlinear equations, non-Gaussian processes.

1. INTRODUCTION

A *flame ball* is an isolated three-dimensional combustion spot with spherical symmetry that occurs in a lean premixed mixture when the combustion process is the one-step irreversible chemical reaction $Fuel \rightarrow products + heat$. In premixed combustion all reactants are intimately mixed at the molecular level before the combustion is started, while in nonpremixed combustion the fuel and the oxidant must be mixed before than combustion can take place. Premixed combustion includes also the familiar laboratory Bunsen burner as well as the flame inside a gasoline-fueled internal combustion engine. Moreover, understanding combustion in lean conditions has a key role in product engineering because it is involved in designing of efficient, clean-burning combustion engines. In fact lean premixed combustion is characterized by low production of NO_x and particulate and then it is of paramount importance to challenge the environmental emergency and to meet future emission standard.

Flame ball was theoretically predicted in 1944 by the Russian physicist Ya. B. Zeldovich (1944) as exact solution to the heat and mass conservation equations in spherical geometry with radial coordinate denoted by r,

$$\rho C_p \left(\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial r} \right) = h \, \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + QW \,, \quad (1a)$$

$$\rho \left(\frac{\partial Y_F}{\partial t} + U \frac{\partial Y_F}{\partial r} \right) = \rho D_F \, \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) - W \,, \quad (1b)$$

where T is the temperature, Y_F the mass fraction of fuel, U the radial velocity, W the chemical rate, Q the heat of reaction, ρ the specific mass, C_p the specific heat at constant pressure, h the heat conductivity, D_F the diffusion coefficient of fuel. Temperature and mass concentration fields are related in (1) by

$$Le \, (T - T_\infty) = \frac{Q}{C_p} \left(1 - \frac{Y_F}{Y_\infty} \right) \,, \quad (2)$$

where the nondimensional number $Le = h/(\rho \, D_F \, C_p)$ is called Lewis number and T_∞ and Y_∞ are the reference values for temperature and mass fraction of fuel, respectively.

After transformation (2), and setting without loss of generality $T_\infty = 0$ and $Y_\infty = 1$, equation for T (1a) gives

$$\rho \left(\frac{\partial Y_F}{\partial t} + U \frac{\partial Y_F}{\partial r} \right) = Le \, \rho D_F \, \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) - LeW . \quad (3)$$

When, for large activation energy, the chemical source term behaves like a Dirac δ-function at the flame sheet (Buckmaster et al., 1990), as it has been clearly reviewed by Ronney et al. (1998), the solutions to steady, convection-free diffusion equations for temperature and chemical species concentration, i.e.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = 0 \quad \text{and} \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) = 0 \,, \quad (4)$$

are of the form $c_1 + c_2/r$, where c_1 and c_2 are constants. This form satisfies the requirement that T and Y_F be bounded as $r \to \infty$. For cylindrical and planar geometry the corresponding forms are $c_1 + c_2 \ln r$ and $c_1 + c_2 r$, respectively, which are obviously unbounded as $r \to \infty$. For this reason theory admits steady flame ball solutions, but not steady "flame cylinder" or steady "flame slab" solutions.

Zeldovich showed that, for an adiabatic flame ball, the temperature at the surface of the flame ball T_* is $T_* = T_{ad}/Le$, where T_{ad} is the adiabatic homogeneous flame temperature. Then, inside the ball ($r < R$), the temperature profile $T(r)$ is constant and corresponds to the combustion product temperature and, outside the ball

$(r > R)$, it decreases depending on the flame ball radius R as

$$T(r) = T_* \frac{R}{r}, \quad r > R. \tag{5}$$

What concerns the fuel mass fraction, it is null inside the ball and, as it follows from formula (2), it increases outside the ball as

$$Y_F = 1 - \frac{R}{r}, \quad r > R, \tag{6}$$

where $C_p T_{ad}/Q = 1$. This steady state can be realized only if the flame ball radius R is constant in time. Then, the evolution equation for the flame ball radius and the later analysis on the stability of the solution are necessary. Here only the derivation of the evolution equation is considered.

Even if theoretically predicted in 1944, stable flame balls were accidentally experimentally discovered only in 1984, during short-duration drop tower experiments conducted by P. D. Ronney and collaborators (Ronney, 1990; Ronney et al., 1994). They were finally experimentally established in 1998 from space flight experiment conducted on the STS-83/MSL-1 Space Shuttle mission (Ronney et al., 1998, see also http://spaceresearch.nasa.gov/research_projects/sts-107_sofball.html). A micro-gravity environment is needed to obtain spherical symmetry and to avoid buoyancy-induced extinction of the flame ball.

Flame balls can exist if $T_* > T_{ad}$ and this condition is met when $Le < 1$, while conventional propagating flames are observed under any value of Lewis number. The reason is that for $T_* < T_{ad}$, $Le > 1$, the flame balls are weaker than plane flames.

Literature models for flame ball radius R are based on non-linear fractional differential equations derived by matching of multiple asymptotic expansions (Joulin, 1985; Buckmaster et al., 1990, 1991; Guyonne and Noble, 2007). The aim of the present study is to proposed a new method to derive the evolution equation of flame ball radius that be more simple than methods proposed in literature. This new simple method can help the advance of research on this topic, in particular on finding analytical and/or numerical solution, its properties, and on stability analysis.

In Section 2 the mathematical preliminaries used in the rest of the text on Fractional Calculus and anomalous diffusion are introduced. In Section 3 the new method is presented in the cases studied by Joulin (1985) and Buckmaster et al. (1990, 1991) and the corresponding equations are derived. In Section 4 the new method is used to formulate an equation for the evolution of the flame ball radius in the general case with a polynomial forcing and anomalous diffusion. Finally in Section 5 the conclusion and perspective for future developments are discussed.

2. MATHEMATICAL PRELIMINARIES

2.1 Fundamentals of Fractional Calculus

This introductory section to Fractional Calculus follows the 1996 CISM lectures by Gorenflo and Mainardi (1997), which were partly based on the book on Abel Integral Equations by Gorenflo and Vessella (1991) and on the article by Gorenflo and Rutman (1995).

Let $f(t)$, with $t > 0$, be a sufficiently well-behaved function, Riemann–Liouville and Caputo fractional derivatives are both based on Riemann-Liouville fractional integral that, when it is of order $\alpha > 0$, is defined as

$$ {}_tJ^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0. \tag{7}$$

The operator ${}_tJ^\alpha$ is conventionally the Identity operator when $\alpha = 0$, i.e. ${}_tJ^0 = I$, and it meets the semigroup property

$$ {}_tJ^\alpha\, {}_tJ^\beta = {}_tJ^\beta\, {}_tJ^\alpha = {}_tJ^{\alpha+\beta}, \quad \alpha, \beta \geq 0. \tag{8}$$

The most simple and useful example of Riemann–Liouville fractional integration is the function $f(t) = t^\nu$, for $t > 0$,

$$ {}_tJ^\alpha\, t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\alpha)} t^{\nu+\alpha}, \quad \alpha \geq 0, \quad \nu > -1. $$

The Riemann-Liouville time fractional derivative of order $\mu > 0$ is defined, in analogy with the ordinary derivative, as the operator ${}_tD^\mu$ which is the left inverse of the Riemann-Liouville integral of order μ

$$ {}_tD^\mu\, {}_tJ^\mu = I, \quad \mu > 0. \tag{9}$$

If m denotes the positive integer such that $m-1 < \mu \leq m$, then from (8) and (9) it follows that ${}_tD^\mu f(t) := {}_tD^m\, {}_tJ^{m-\mu} f(t)$. Hence for $m-1 < \mu < m$

$$ {}_tD^\mu f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\mu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\mu+1-m}}\, d\tau \right], \tag{10}$$

and ${}_tD^\mu f(t) = d^m f(t)/dt^m$ when $\mu = m$.

On the other hand, the fractional derivative of order $\mu > 0$ in the Caputo sense is defined as the operator ${}_tD_*^\mu$ such that ${}_tD_*^\mu f(t) := {}_tJ^{m-\mu}\, {}_tD^m f(t)$. Hence for $m-1 < \mu < m$

$$ {}_tD_*^\mu f(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\mu+1-m}}\, d\tau, \tag{11}$$

and ${}_tD_*^\mu f(t) = d^m f(t)/dt^m$ when $\mu = m$. Thus, when the order is not integer the two fractional derivatives mainly differ because the derivative of order m does not generally commute with the fractional integral.

Furthermore, unlike Riemann–Liouville fractional derivative, Caputo fractional derivative satisfies the relevant property of being zero when it is applied to a constant, and, in general, when its order μ is such that $m-1 < \mu \leq m$, to any power function of non-negative integer degree less than m. Indeed, what concerns Riemann–Liouville derivative operator, for $t > 0$,

$$ {}_tD^\mu\, t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\mu)} t^{\nu-\mu}, \quad \mu \geq 0, \quad \nu > -1. \tag{12}$$

Finally, for $m-1 < \mu \leq m$,

$$ {}_tD^\mu f(t) = {}_tD^\mu g(t) \text{ if } f(t) = g(t) + \sum_{j=1}^m k_j\, t^{\mu-j}, \tag{13}$$

$$ {}_tD_*^\mu f(t) = {}_tD_*^\mu g(t) \text{ if } f(t) = g(t) + \sum_{j=1}^m k_j\, t^{m-j}, \tag{14}$$

where coefficients k_j are arbitrary constants.

Gorenflo and Mainardi (1997) have shown the essential relationship between the two fractional derivatives (when both of them exist), for $m-1<\mu<m$, which is

$$ {}_tD_*^{\mu} f(t) = {}_tD^{\mu}\left[f(t) - \sum_{n=0}^{m-1} f^{(n)}(0^+)\,\frac{t^n}{n!}\right] , \tag{15}$$

and applying (12)

$$ {}_tD_*^{\mu} f(t) = {}_tD^{\mu} f(t) - \sum_{n=0}^{m-1} \frac{f^{(n)}(0^+)\,t^{n-\mu}}{\Gamma(n-\mu+1)} . \tag{16}$$

In the special case $f^{(n)}(0^+)=0$ for $n=0,1,\ldots,m-1$, the identity between the two fractional derivatives follows.

Caputo fractional derivative is a regularization in the time origin of the Riemann-Liouville fractional derivative. From (16) emerges that for its existence all the limiting values $f^{(n)}(0^+) := \lim_{t\to 0^+} f(t)$ are required to be finite for $n=0,1,2,\ldots,m-1$.

To conclude here it is highlighted the different behaviour of the two fractional derivatives at the end points of the interval $(m-1,m)$, i.e. when the order is any positive integer,

$$\begin{cases} \lim\limits_{\mu\to(m-1)^+} {}_tD_*^{\mu} f(t) = f^{(m-1)}(t) - f^{(m-1)}(0^+) , \\ \\ \lim\limits_{\mu\to m^-} {}_tD_*^{\mu} f(t) = f^{(m)}(t) , \end{cases} \tag{17}$$

so whereas ${}_tD^{\mu}$ is, with respect to its order μ, an operator continuous at any positive integer, ${}_tD_*^{\mu}$ is an operator left-continuous.

2.2 Anomalous diffusion modelling

A typical diffusion process is a process described by the classical diffusion equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2} , \quad x\in\mathcal{R} , \quad t\in\mathcal{R}_0^+ , \tag{18}$$

with $P(x,0)=\delta(x)$, where $P(x,t)$ is the probability density function to find a particle in x at time t. The solution of (18) is the Gaussian function and the displacement variance grows linearly with time, i.e $\langle x^2\rangle \sim t$. This type of diffusion is also referred to as normal diffusion, to distinguish it from anomalous diffusion in which the displacement variance grows nonlinearly in time, for example with the power law $\langle x^2\rangle \sim t^{\beta}$. Generally, anomalous diffusion is met in complex media. Obviously the classical diffusion equation (18) is not more correct for describing anomalous diffusion processes (Klafter and Sokolov, 2005).

In literature, anomalous diffusion is modelled in several different ways, however Fractional Calculus turns out to be one of the most successful tool (Sokolov et al., 2002). Models based on fractional differential equations have been proposed in a large number of research fields. In this respect the valuable work by Prof. R Gorenflo is here noticed and remarked by reminding some of his most cited papers: (Gorenflo and Mainardi, 1998; Gorenflo et al., 2000, 2002).

The main characteristic that relates fractional differential equations to anomalous diffusion is that, when the solution is interpreted as probability density function, the particle displacement variance turns out to be driven by the fractional order of derivation. The most simple example of anomalous diffusion described by fractional differential equations is the time-fractional diffusion equation (Mainardi and Pagnini, 2003)

$$ {}_tD_*^{\beta} P(x,t) = \frac{\partial^2 P}{\partial x^2} , \quad x\in\mathcal{R} , \quad t\in\mathcal{R}_0^+ , \tag{19}$$

with $0<\beta\le 2$ and initial condition $P(x,0)=P_0(x)$, which gives a variance driven by the power law $\langle x^2\rangle \sim t^{\beta}$.

In general, anomalous diffusion corresponding to different phenomena is described by different fractional differential equations which could be also nonlinear, see e.g. (Lenzi et al., 2009). However, in all cases, if the process is self-similar and the variance is proportional to t^{β} then the probability density function of particle position has the general form

$$P(x,t) = \frac{1}{t^{\beta/2}}\,\mathcal{P}\left(\frac{x}{t^{\beta/2}}\right) . \tag{20}$$

In fact

$$\begin{aligned} \langle x^2\rangle = \int_{-\infty}^{+\infty} x^2 P(x,t)\,dx &= \int_{-\infty}^{+\infty} \frac{x^2}{t^{\beta/2}}\mathcal{P}\left(\frac{x}{t^{\beta/2}}\right)\,dx \\ &= \left\{ \int_{-\infty}^{+\infty} \xi^2 \mathcal{P}(\xi)\,d\xi \right\}\cdot t^{\beta} \\ &= const\cdot t^{\beta} , \end{aligned}$$

after application of the change of variable $x=\xi\,t^{\beta/2}$ and provided that $\int_{-\infty}^{+\infty} \xi^2\mathcal{P}(\xi)\,d\xi<\infty$.

3. EVOLUTION OF FLAME BALL RADIUS

3.1 Description of the method

Let R be at any fixed instant t the radius of the flame ball, then its growing in time is here assumed to be determined by the evolution of the matching interface between an inner kernel ($r<R$), which is the quasi-stationary spherical solution of a Poisson-type equation, and an outer diffusive part ($r>R$), which is the solution of a diffusion equation.

Let ϕ_s be the inner solution and ϕ_d be the outer solution. Then the growing in time of the flame ball radius is determined by a diffusion operator that acts on the inner solution computed on the surface of the flame ball. This means that the source term of the diffusion process is determined by $\phi_s(x,t)\delta(x-R(t))$ and the action of the operator emerges to be a double convolution integral both in space and time with propagating kernel $\mathcal{K}(x,t)$, i.e.

$$R(t) = \mathcal{K}(x,t) * \phi_s(x,t)\delta(x-R(t)) = \phi_d(R,t) . \tag{21}$$

This matching method has been suggested to the author by the diffusive formulation discussed in (Audounet et al., 1998; Audounet and Roquejoffre, 1998; Rouzaud, 2001). Moreover, such diffusive formulation, has been used by

Gorenflo and Vessella (1991) to study Volterra integral equations.

3.2 The inner solution

Consider a flame initiated by a point source energy input, which spherically evolves under the action of a radial forcing $\sim 1/r^2$ and radiative heat losses $\sim -\lambda$. Then the inner solution in spherical coordinates $\phi_s(r,t)$ is determined as the quasi-stationary solution of the Poisson-type equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \phi_s}{\partial r}\right] = \frac{2}{r^2} - 12\lambda\,, \tag{22}$$

with boundary condition

$$\left[r^2\frac{\partial \phi_s}{\partial r}\right]_{r=0} = -2Eq(t)\,, \quad q(0)=0\,, \tag{23}$$

where $Eq(t)$ is a measure of the energy input with $E>0$ as intensity and $q(t)$ as temporal variation. The numerical factors on RHS of (22) and (23) are chosen for formal reasons. Finally, the inner solution $\phi_s(r,t)$ turns out to be

$$\phi_s(r,t) = 2\left[\ln r + \frac{E\,q(t)}{r} - \lambda\,r^2\right] = 2\,f(r,t)\,. \tag{24}$$

3.3 The outer solution

Each point of the matching interface is assumed to be diffused along the one-dimensional axes that ranges from $-\infty$ to $+\infty$ and is aligned with r. Then, the spherical reference system characterized by $r>0$, which was used to determine the growing of the inner solution ϕ_s, is now abandoned to use a one-dimensional Cartesian axes x, such that $|x|=r$, and the diffusion is modelled with respect this reference frame. This means that now the flame ball radius is located in $|x|=R$.

Finally, the outer diffusive solution ϕ_d is determined as the solution of a diffusion equation with source term S given by the inner solution computed in the inner-outer interface located at the flame position $R(t)$. In the r-coordinate system $S(r,t)=\phi_s(r,t)\delta(r-R(t)) = 2f(r,t)\delta(r-R(t))$ and in the x-coordinate system $S(x,t) = \phi_s(x,t)\delta(x-R(t)) = 2f(x,t)\delta(x-R(t))$. Hence ϕ_d is the solution of the diffusion equation

$$\frac{\partial \phi_d}{\partial t} = \frac{\partial^2 \phi_d}{\partial x^2} + 2\,f(x,t)\,\delta(x-R(t))\,. \tag{25}$$

The Green function of (25) is the Gaussian density

$$\mathcal{G}(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{x^2}{4t}\right\}\,, \tag{26}$$

which describes a normal diffusion process with linear variance growing, i.e. $\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2\mathcal{G}(x,t)\,dx \sim t$. Then the solution of (25) turns out to be the following double convolution integral

$$\phi_d(x,t) = 2\int_{-\infty}^{+\infty} d\xi \int_0^t d\tau\,\mathcal{G}(x-\xi,t-\tau)f(\xi,\tau)\delta(\xi-R(\tau))\,,$$

which solving the convolution in space reduces to

$$\phi_d(x,t) = 2\int_0^t \mathcal{G}(x-R(\tau),t-\tau)f(\xi,\tau)\,d\tau\,.$$

To conclude, inserting (26) in the above formula, the solution of (25) turns out to be

$$\phi_d(x,t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\mathrm{e}^{-(x-R)^2/(4(t-\tau))}}{\sqrt{t-\tau}}\,f(R(\tau),\tau)\,d\tau\,. \tag{27}$$

3.4 The evolution equation

Comparing (21) and (27) it emerges that the propagator $\mathcal{K}(x,t)$ turns out to be the Gaussian density (26) and the evolution equation for the flame ball radius follows to be

$$R(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{f(R(\tau),\tau)}{\sqrt{t-\tau}}\,d\tau = {}_tJ^{1/2}[f(R(t),t)]\,, \tag{28}$$

with initial condition $R(0)=0$, where ${}_tJ^{1/2}$ is the Riemann–Liouville fractional integral of order $1/2$ defined in (7). Applying the Riemann–Liouville time-fractional derivative operator ${}_tD^{1/2}$, which is defined in (10), on both sides of (28) gives ${}_tD^{1/2}R(t) = {}_tD^{1/2}{}_tJ^{1/2}[f(R(t),t)] = f(R(t),t)$, where property (9) is used. After multiplication by $R(t)$, the evolution equation (28) becomes the following nonlinear fractional differential equation

$$R(t)\,{}_tD^{1/2}R(t) = R(t)\ln R(t) + Eq(t) - \lambda R^3(t)\,. \tag{29}$$

Relationship (16) between Riemann–Liouville ${}_tD^{\mu}$ and Caputo ${}_tD_*^{\mu}$ fractional derivatives can be applied in (29). Since the order of fractional derivation is $0<1/2<1$ and $R(0^+)=0$, then ${}_tD^{1/2}R(t) = {}_tD_*^{1/2}R(t)$. Finally, in terms of Caputo time-fractional derivative, the evolution equation of the flame radius is

$$R(t)\,{}_tD_*^{1/2}R(t) = R(t)\ln R(t) + Eq(t) - \lambda R^3(t)\,. \tag{30}$$

Equation (30) is the Buckmaster–Joulin–Ronney equation (Buckmaster et al., 1990, 1991) and it reduces to the seminal equation derived by Joulin (1985) neglecting heat losses, i.e. $\lambda=0$.

The problem of stability of the flame ball is important to theoretically design the experimental realization of the stable flame balls predicted by Zeldovich, but also for applicative reasons, to maintain the combustion in the most efficient regime and to prevent the quenching of the flame, and for security reason, to avoid that the radius diverges. Resuming literature results, here it is briefly reminded that when radiative heat losses are larger than a critical value, i.e. $\lambda>\lambda_{cr}$, then the flame always quenches; otherwise when $\lambda<\lambda_{cr}$ the flame quenches if $E<E_{cr}(q)$ and it stabilizes to R_2 (or R_1) if $E>E_{cr}(q)$ (or $E=E_{cr}(q)$), where $R_2>R_1$ are the solutions of the equation $\ln R = \lambda R^2$. For more details on stability of solution of (30), the interest reader is referred to (Rouzaud, 2001, 2003; Roquejoffre and Rouzaud, 2006) and to (Joulin, 1985; Audounet et al., 1998; Lederman et al., 2002) for the analysis of solution of the original Joulin equation without heat losses.

Moreover, it is here also reminded that numerical solution of (30) is not a fully solved task. This problem is addressed in (Audounet et al., 2002), or in (Audounet and Roquejoffre, 1998; Dubois and Mengué, 2003; Diethelm and Weilbeer, 2004) for Joulin equation.

4. GENERALIZED EVOLUTION OF FLAME BALL RADIUS

4.1 Generalized inner solution

Here it is called generalized inner solution $\Phi_s(r,t)$ the spherical solution of the following Poisson-type equation with a general polynomial forcing

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \Phi_s}{\partial r}\right] = \frac{\mathcal{N}}{r^2} - \mathcal{N}\,\lambda \sum_{i=1}^{n} 6\gamma_i\, r^{\alpha_i}\,, \tag{31}$$

with the boundary condition

$$\left[r^2\frac{\partial \Phi_s}{\partial r}\right]_{r=0} = -\mathcal{N}\,Eq(t)\,, \quad q(0)=0\,, \tag{32}$$

where the coefficients in the RHS of (31) and (32) are chosen for the same formal reasons of coefficients in (22) and (23). Then the generalized inner solution $\Phi_s(r,t)$ is

$$\begin{aligned}\Phi_s(r,t) &= \mathcal{N}\left[\ln r + \frac{E\,q(t)}{r} - \lambda r^2 \sum_{i=1}^{n} \frac{6\gamma_i\, r^{\alpha_i}}{(\alpha_i+3)(\alpha_i+2)}\right]\,,\\ &= \mathcal{N}\,F(r,t)\,. \end{aligned}\tag{33}$$

For $n=1$ and setting $\mathcal{N}=2$, $\gamma_1=1$, $\alpha_1=0$, previous inner solution (24) is recovered as well as $F(r,t)=f(r,t)$.

4.2 Generalized outer solution

As explained in §3.3, also in this case the diffusion process occurs along the axes x aligned with $r=|x|$. So the description of the process moves from the spherical coordinate r to a one-dimensional Cartesian reference frame. Anomalous diffusion is characterized by a nonlinear growing rate in time of the variance, here the following power law is considered: $\langle x^2\rangle \sim t^\beta$, $\beta>0$.

In §2.2 it has been pointed out that different types of anomalous diffusion equation have been proposed in literature. However, from all evolution equations to model anomalous diffusion which admit a self-similar solution, Green function emerges to be of the form (20)

$$\mathcal{G}_\beta(x,t) = \frac{1}{t^{\beta/2}}\,\mathcal{H}\left(\frac{x}{t^{\beta/2}}\right)\,. \tag{34}$$

Then the solution of the whole diffusion process with source term $S(x,t)=\Phi_s(x,t)\delta(x-R(t)) = \mathcal{N}F(x,t)\delta(x-R(t))$ is given by the double convolution integral

$$\Phi_d(x,t) = \mathcal{N}\int_{-\infty}^{+\infty} d\xi \int_0^t d\tau\, \mathcal{G}_\beta(x-\xi,t-\tau) f(\xi,\tau)\delta(\xi-R(\tau))\,,$$

which solving the convolution in space reduces to

$$\Phi_d(x,t) = \mathcal{N}\int_0^t \mathcal{G}_\beta(x-R(\tau),t-\tau) f(\xi,\tau)\,d\tau\,.$$

To conclude, inserting (34) in the above formula, the generalized outer solution turns out to be

$$\Phi_d(x,t) = \mathcal{N}\int_0^t \mathcal{H}\left[\frac{x-R(t)}{(t-\tau)^{\beta/2}}\right]\frac{F(R(\tau),\tau)}{(t-\tau)^{\beta/2}}\,d\tau\,. \tag{35}$$

4.3 Generalized evolution equation

The generalized evolution equation follows from (21) and (35) from which the propagator $\mathcal{K}(x,t)$ is given by the Green function (34). Setting $\mathcal{H}(0)=1/(\mathcal{N}\,\Gamma(1-\beta/2))$, it turns out to be

$$\begin{aligned}R(t) &= \frac{1}{\Gamma(1-\beta/2)}\int_0^t \frac{F(R(\tau),\tau)}{(t-\tau)^{\beta/2}}\,d\tau\\ &= {}_tJ^{1-\beta/2}[F(R(t),t)]\,. \end{aligned}\tag{36}$$

Repeating the same steps as for non anomalous case, in terms of Riemann–Liouville fractional differential operator equation (36) becomes

$$\begin{aligned}R(t)\,D_t^{1-\beta/2}R(t) &= R(t)\ln R(t) + Eq(t)\\ &\quad -\lambda R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i\,R^{\alpha_i}(t)}{(\alpha_i+3)(\alpha_i+2)}\,, \end{aligned}\tag{37}$$

and in terms of Caputo fractional differential operator

$$\begin{aligned}R(t)\,{}_*D_t^{1-\beta/2}R(t) &= R(t)\ln R(t) + Eq(t)\\ &\quad -\lambda R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i\,R^{\alpha_i}(t)}{(\alpha_i+3)(\alpha_i+2)}\,. \end{aligned}\tag{38}$$

About stability and threshold phenomenon of (38), the case when the logarithmic function is omitted is considered in (Rouzaud, 2003, §5).

For normal diffusion (i.e. $\beta=1$) and $n=1$, if $\gamma_1=1$ and $\alpha_1=0$ then the generalized evolution equation (38) reduces to Buckmaster–Joulin–Ronney equation (30), and to Joulin equation setting also $\lambda=0$; if $\gamma_1=1$ and $\alpha_1=-1$ it reduces to

$$R(t)\,{}_*D_t^{1/2}R(t) = R(t)\ln R(t) + Eq(t) - 3\lambda R^2(t)\,, \tag{39}$$

which has been derived by Guyonne and Noble (2007) on the basis of the linearized Eddington equation for radiative field.

5. CONCLUSION

In the present paper the problem of a flame ball is addressed, in particular, it has been considered the derivation method of the evolution equation for the flame ball radius.

Even if some evolution equations have been derived in literature, this is not a fully established issue. In this respect here a new method to derive the evolution equation of the radius of a flame ball is proposed. This method is based on the idea to split the flame ball in two components: the inner kernel, which is driven by a Poisson-type equation with a general polynomial forcing term, and the outer part, which is driven by a generalized anomalous diffusion process. The evolution equation for the radius of the flame ball is determined as the evolution equation for the interface that matches the solution of the inner spherical kernel and the solution of the outer diffusive part. The resulting equation turns out to be a nonlinear fractional differential equation whose fractional order of derivation emerges to be related to the diffusion process. In fact, the exponent of the power law of displacement variance $\langle x^2\rangle \sim t^\beta$ drives the order of fractional derivation which turns out to be $1 - \beta/2$ and it reduces to $1/2$ when the diffusion process is Gaussian.

This method strongly simplifies and generalizes previous derivations. In fact since a polynomial forcing and anomalous diffusion are considered, literature equations (Joulin, 1985; Buckmaster et al., 1990; Guyonne and Noble, 2007) are recovered when the forcing and the diffusion process are appropriately chosen.

The main remarkable aspect of this new method is that, due to its clear and simple derivation, it can be a useful tool to further development and advance in the research on this topic helping to overcame the difficulties that the current models meet. In fact, the mathematical simplicity of equation foundation can highlight new promising way to find analytical and numerical solution, solution properties as well as to analyses solution stability which is of paramount importance for establishing the experimental configuration to observe the steady flame ball predicted by Zeldovich.

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