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SHORT NOTE

ERDÉLYI–KOBER FRACTIONAL DIFFUSION

Gianni Pagnini

Abstract

The aim of this Short Note is to highlight that the *generalized grey Brownian motion* (ggBm) is an anomalous diffusion process driven by a fractional integral equation in the sense of Erdélyi–Kober, and for this reason here it is proposed to call such family of diffusive processes as *Erdélyi–Kober fractional diffusion*. The ggBm is a parametric class of stochastic processes that provides models for both fast and slow anomalous diffusion. This class is made up of self-similar processes with stationary increments and it depends on two real parameters: $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. It includes the fractional Brownian motion when $0 < \alpha \leq 2$ and $\beta = 1$, the time-fractional diffusion stochastic processes when $0 < \alpha = \beta < 1$, and the standard Brownian motion when $\alpha = \beta = 1$. In the ggBm framework, the Mainardi function emerges as a natural generalization of the Gaussian distribution recovering the same key role of the Gaussian density for the standard and the fractional Brownian motion.

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Key Words and Phrases: anomalous diffusion, Erdélyi–Kober fractional integral and derivative, Mainardi function

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Diffusion in disordered media is commonly named *anomalous*, to mark a difference with the classical diffusion processes where the probability density function *pdf* to find a particle in the place x at the time t is Normal,

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i.e., Gaussian, and for this it is called *Normal* diffusion, or *Gaussian* diffusion. Moreover, beside the non-Gaussian particle *pdf*, the anomalously is embodied even by the non-linear growing in time of the variance of the particle spreading [8]. The process of anomalous diffusion is referred to as *fast diffusion*, when the variance of the particle spreading grows with a power law with exponent greater than 1, and it is referred to as *slow diffusion*, when such exponent is lower than 1.

In the case of anomalous diffusion, the classical local flux-gradient relationship does not hold and it is necessary to determine a non-local relationship. It is well-known that a useful mathematical tool for physical investigation and description of non-local and anomalous diffusion is Fractional Calculus, which is that branch of mathematical analysis dealing with pseudo-differential operators interpreted as integrals and derivatives of non-integer order [7, 26].

Non-locality can be designated in time (time-fractional diffusion) or in space (space-fractional diffusion), as well as both in space and time (space-time fractional diffusion equation) [15]. Generally, when the fractional differentiation is considered for the time, then the fractional derivative operator is assumed to be in the Caputo or in the Riemann–Liouville sense, when the fractional differentiation is considered for the space, then the fractional derivative operator is assumed to be in the Riesz–Feller sense.

Recently, the extension of fractional differential equations to distributed-order fractional differential equations has permitted to describe also processes whose scaling law changes in time, see e.g. [17, 28, 35].

Furthermore, under the physical point of view, when there is no separation of timescale between the microscopic and the macroscopic level of the process the randomness of the microscopic level is transmitted to the macroscopic level and the correct description of the macroscopic dynamics has to be in terms of the Fractional Calculus [6]. Moreover, fractional integro/differential equations are related to phenomena with fractal properties [27].

A fractional differential approach has been successfully used for modelling in several different disciplines as for example statistical physics [19], neuroscience [12], economy [29], control theory [37] and combustion science [24]. Further applications of the fractional approach are recently introduced and discussed by J.A. Tenreiro Machado [36].

Normal diffusion, or Gaussian diffusion, is a Markovian stochastic process driven by the classical parabolic equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2}\,, \quad x \in \mathcal{R}\,, \quad t \in \mathcal{R}_0^+\,, \tag{1}$$

with initial condition $P(x,0) = P_0(x)$. The fundamental solution of (1), which is named also Green function, and corresponding to the case with initial condition $P(x,0) = P_0(x) = \delta(x)$, is the Gaussian density

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, \tag{2}$$

whose variance grows linearly in time, i.e., $\langle x^2 \rangle = \displaystyle\int_{-\infty}^{+\infty} x^2 f(x,t)\, dx = 2\,t$. The density function $P(x,t)$ with general initial condition $P(x,0) = P_0(x)$ is related to the fundamental solution $f(x,t)$ by the following convolution integral

$$P(x,t) = \int_{-\infty}^{+\infty} f(\xi,t) P_0(x-\xi)\, d\xi\,. \tag{3}$$

In order to generalize the classical Markovian setting to Non-Markovian cases, the following integral equation has been introduced by Mura, Taqqu and Mainardi [23]:

$$P(x,t) = P_0(x) + \int_0^t \frac{\partial g(s)}{\partial s} K[g(t)-g(s)] \frac{\partial^2 P(x,s)}{\partial x^2}\, ds\,, \tag{4}$$

where $K(t)$ is a memory kernel and $g(t)$, with $g(0) = 0$, is a smooth and increasing function describing a time stretching. The Green function of (4) $\mathcal{G}(x,t)$, which is the marginal one-point one-time *pdf* of the non-Markovian diffusion process, turns out to be

$$\mathcal{G}(x,t) = \int_0^{\infty} f(x,\tau) h(\tau, g(t))\, d\tau\,, \tag{5}$$

where $f(x,t)$ is the Gaussian density (2) that is the fundamental solution of the Markovian diffusion process, i.e., $K(t) = \delta(t)$, and $h(\tau,t)$ is the fundamental solution of the so-called *non-Markovian forward drift equation*

$$u(\tau,t) = u_0(\tau) - \int_0^t K(t-s) \frac{\partial u(\tau,s)}{\partial \tau}\, ds\,, \quad \tau\,, t \in \mathcal{R}_0^+\,, \tag{6}$$

where $u_0(\tau) = u(\tau,0)$.

When the kernel and the time-stretching functions are stated as

$$K(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}\,, \quad g(t) = t^{\alpha/\beta}\,, \quad 0 < \alpha \le 2\,, \quad 0 < \beta \le 1\,, \tag{7}$$

Equation (4) becomes

$$P(x,t) = P_0(x) + \frac{1}{\Gamma(\beta)} \frac{\alpha}{\beta} \int_0^t \tau^{\alpha/\beta - 1} \, (t^{\alpha/\beta} - \tau^{\alpha/\beta})^{\beta-1} \, \frac{\partial^2 P(x,\tau)}{\partial x^2}\, d\tau\,, \tag{8}$$

that was originally introduced by A. Mura in his PhD Thesis [20], and later discussed by him and collaborators in a number of papers [16, 21, 22, 23].

It is well-known that it exists a relationship between the solutions of a certain class of integral equations that are used to model anomalous diffusion and stochastic processes. In this respect, the density function $P(x,t)$ which solves (8) is the marginal particle *pdf*, i.e., the one-point one-time density function of particle dispersion, of the *generalized grey Brownian motion* (ggBm) [20, 21, 22].

The ggBm is a special class of H-sssi processes of order $H = \alpha$, or Hurst exponent $H = \alpha/2$, where, according to a common terminology, H-sssi means H-self-similar-stationary-increments. The ggBm provides non-Markovian stochastic models for anomalous diffusion, of both slow type $0 < \alpha < 1$ and fast type $1 < \alpha < 2$. The ggBm includes some well-known processes, so that it defines an interesting general theoretical framework. In fact, the fractional Brownian motion appears for $\beta = 1$, the grey Brownian motion, in the sense of W. R. Schneider [30, 31], corresponds to the choice $0 < \alpha = \beta < 1$, and finally the standard Brownian motion is recovered by setting $\alpha = \beta = 1$. It is worth noting to remark that only in the particular case of the Brownian motion the stochastic process is Markovian. Moreover, the ggBm is not an ergodic process [22].

The integral in the non-Markovian kinetic equation (8) can be expressed in terms of an Erdélyi–Kober fractional integral. In fact, let μ, η and γ be $\mu > 0$, $\eta > 0$ and $\gamma \in \mathcal{R}$, *the Erdélyi–Kober fractional integral operator* $I_\eta^{\gamma,\mu}$, for a sufficiently well-behaved function $\varphi(t)$, is defined as [7, formula (1.1.17)]

$$\begin{aligned} I_\eta^{\gamma,\mu}\,\varphi(t) &= \frac{t^{-\eta(\mu+\gamma)}}{\Gamma(\mu)} \int_0^t \tau^{\eta\gamma}\,(t^\eta - \tau^\eta)^{\mu-1} \varphi(\tau)\,d(\tau^\eta) \\ &= \frac{\eta}{\Gamma(\mu)} t^{-\eta(\mu+\gamma)} \int_0^t \tau^{\eta(\gamma+1)-1} (t^\eta - \tau^\eta)^{\mu-1} \varphi(\tau)\,d\tau\,, \end{aligned} \tag{9}$$

hence equation (8) can be re-written as

$$P(x,t) = P_0(x) + t^\alpha \left[I_{\alpha/\beta}^{0,\beta} \frac{\partial^2 P}{\partial x^2} \right] . \tag{10}$$

The integral operator $I_\eta^{\gamma,\mu}$ was introduced by I.N. Sneddon (see for example [32, 33, 34]) who studied its basic properties and emphasized its useful applications to the generalized axially symmetric potential theory (GASPT) and other physical problems (say in electrostatics, elasticity, etc). When $\eta = 1$, one obtains the operators of fractional integration as originally introduced by H. Kober [9] and A. Erdélyi [1] and, when $\eta = 2$, those introduced by I.N. Sneddon [32, 33, 34]. In the special case $\gamma = 0$ and $\eta = 1$, the Erdélyi–Kober fractional integral operator (9) and the Riemann–Liouville fractional integral of order μ, here noted by J^μ, are related by the formula

$$I_1^{0,\mu}\,\varphi(t) = \frac{t^{-\mu}}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\varphi(\tau)\,d\tau = t^{-\mu}J^{\mu}\,\varphi(t)\,. \tag{11}$$

The above remark about the relationship between the governing equation of the ggBm (8) and the Erdélyi–Kober operator of Fractional Calculus (9) constitutes the aim of this Short Note. The possibility to re-write equation (8) as (10) was briefly noted by the author in [25]. However, here this correspondence between the ggBm and the Erdélyi–Kober fractional integral operator is stressed and, since the ggBm serves as a stochastic model for the anomalous diffusion, this leads to define the family of diffusive processes governed by the ggBm as *Erdélyi–Kober fractional diffusion*.

In order to establish the diffusion-type equation corresponding to (8), we need also the notion of the Erdélyi-Kober fractional differential operator. Let $n-1<\mu\le n$, $n\in\mathcal{N}$, *the Erdélyi–Kober fractional derivative* is defined as [7, formula (1.5.19)]

$$D_\eta^{\gamma,\mu}\,\varphi(t) = \prod_{j=1}^{n}\left(\gamma+j+\frac{1}{\eta}t\frac{d}{dt}\right)(I_\eta^{\gamma+\mu,n-\mu}\,\varphi(t))\,. \tag{12}$$

The Riemann–Liouville fractional derivative of order μ, $m-1<\mu\le m$, $m\in\mathcal{N}$ is defined as $D_{RL}^{\mu}\,\varphi(t) = \dfrac{d^m}{dt^m}J^{m-\mu}\,\varphi(t)$, and it emerges that the Erdélyi–Kober and the Riemann–Liouville fractional derivatives are related through the formula

$$D_1^{-\mu,\mu}\,\varphi(t) = t^{\mu}D_{RL}^{\mu}\,\varphi(t)\,. \tag{13}$$

A further important property of the Erdélyi–Kober fractional derivative is its reduction to the identity operator when $\mu=0$, i.e.,

$$D_\eta^{\gamma,0}\varphi(t) = \varphi(t)\,. \tag{14}$$

Recently, the notions of the Erdélyi–Kober fractional integrals and derivatives have been further extended by Yu. Luchko [10] and Yu. Luchko & J. Trujillo [11].

Equation (10) in diffusive form is obtained by deriving in time both sides and it results

$$\begin{aligned}\frac{\partial P}{\partial t} &= \alpha\,t^{\alpha-1}I_{\alpha/\beta}^{0,\beta}\frac{\partial^2 P}{\partial x^2} + t^{\alpha}\frac{\partial}{\partial t}\left(I_{\alpha/\beta}^{0,\beta}\frac{\partial^2 P}{\partial x^2}\right)\\ &= t^{\alpha-1}\left[\alpha+t\frac{\partial}{\partial t}\right]\left(I_{\alpha/\beta}^{0,\beta}\frac{\partial^2 P}{\partial x^2}\right)\,,\end{aligned} \tag{15}$$

that can be recast as

$$\frac{\partial P}{\partial t} = \frac{\alpha}{\beta} t^{\alpha-1} \left[(\beta - 1) + 1 + \frac{\beta}{\alpha} t \frac{\partial}{\partial t} \right] \left(I^{0,\beta}_{\alpha/\beta} \frac{\partial^2 P}{\partial x^2} \right) , \tag{16}$$

and finally, by using (12),

$$\frac{\partial P}{\partial t} = \frac{\alpha}{\beta} t^{\alpha-1} \, D^{\beta-1,1-\beta}_{\alpha/\beta} \, \frac{\partial^2 P}{\partial x^2} \, . \tag{17}$$

A diffusion-type equation for the ggBm was previously proposed [16] but adopting, with an abuse of notation, the Riemann–Liouville fractional differential operator with a stretched time variable. Then, since the Erdélyi–Kober fractional differential operator is taken into account, Equation (17) follows to be the correct formulation.

The Green function corresponding to (10, 17) is [20, 21, 22, 23]

$$\mathcal{G}(x,t) = \frac{1}{2} \frac{1}{t^{\alpha/2}} M_{\beta/2} \left(\frac{|x|}{t^{\alpha/2}} \right) , \tag{18}$$

where $M_\nu(z)$ is the M-Wright function, often referred to as Mainardi function in the literature devoted to fractional diffusion [14, 26], and it is defined as [13]

$$\begin{aligned} M_\nu(z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \, \Gamma[-\nu n + (1-\nu)]} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \, \Gamma(\nu n) \sin(\pi \nu n) \, , \quad 0 < \nu < 1 \, , \end{aligned} \tag{19}$$

see Reference [2, 3, 16] for a review. Here it is reminded the noteworthy composition, or subordination-type, formula [18]

$$t^{-\nu} M_\nu \left(\frac{\xi}{t^\nu} \right) = t^{-\ell} \int_0^\infty M_\lambda \left(\frac{\xi}{\tau^\lambda} \right) M_\ell \left(\frac{\tau}{t^\ell} \right) \frac{d\tau}{\tau^\lambda} \, , \quad \text{with} \quad \nu = \lambda \, \ell \, , \tag{20}$$

where $0 < \nu \, , \lambda \, , \ell < 1$ and $\xi \, , t \, , \tau \in \mathcal{R}^+_0$. By using (20) and the special case $M_{1/2}(z) = (1/\sqrt{\pi}) \exp(-z^2/4)$, Green function (18) can be expressed as [16, 22, 23]

$$\mathcal{G}(x,t) = \frac{1}{2} \frac{1}{t^{\alpha/2}} M_{\beta/2} \left(\frac{|x|}{t^{\alpha/2}} \right) \tag{21}$$

$$= \frac{1}{\sqrt{4 t^\alpha}} \int_0^\infty M_{1/2} \left(\frac{|x| t^{-\alpha/2}}{\tau^{1/2}} \right) M_\beta(\tau) \, d\tau \tag{22}$$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi \tau t^\alpha}} \exp \left\{ -\frac{x^2}{4 \tau t^\alpha} \right\} M_\beta(\tau) \, d\tau \, , \tag{23}$$

so that, under the view point of statistical mechanics, the ggBm, or the *Erdélyi–Kober fractional diffusion*, emerges to be the superposition of processes with stretched Gaussian density $\dfrac{1}{\sqrt{4\pi\tau t^\alpha}}\exp\left\{-\dfrac{x^2}{4\tau t^\alpha}\right\}$, i.e. fractional Brownian motions, whose variance is $\langle x^2\rangle = 2\tau t^\alpha$ where τ is a random coefficient distributed according to $M_\beta(\tau)$.

However, equation (23) can be further re-managed to exhibit a subordination type representation. In fact, after the change of variable $t_* = \tau t^\alpha$, it follows that

$$\mathcal{G}(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t_*}}\exp\left\{-\frac{x^2}{4t_*}\right\}\frac{1}{t^\alpha}M_\beta\left(\frac{t_*}{t^\alpha}\right)dt_*\,, \tag{24}$$

which means that the random trajectory $x = x(t)$ can be obtained as a subordination process by $x = x(t) = y[t_*(t)]$, where $t_* = t_*(t)$ is a positive random variable that evolves in the natural time t and it is referred to as operational time [4, 5]. The process $t_* = t_*(t)$ is the directing process that realizes in the $(t,\, t_*)$-plane whose *pdf* is $t^{-\alpha}M_\beta(t_*t^{-\alpha})$, please note that the *pdf* of the directing process belongs to the same family of the Green function $\mathcal{G}(x,t)$ and they differ for the parameter pair, and $y = y(t_*)$ is the parent process that is a random trajectory in the $(t_*,\, y)$-plane with Gaussian *pdf* evolving in the operational time t_*. Geometrically, identifying the spatial coordinates y and x, the subordination structure $x = x(t) = y[t_*(t)]$ is obtained by concatenation.

The marginal *pdf* of the non-Markovian diffusion process ggBm emerges to be related to the Mainardi function M_ν and it describes both slow and fast anomalous diffusion. In fact, the variance of Green function (18) is $\langle x^2\rangle = \displaystyle\int_{-\infty}^{+\infty} x^2\mathcal{G}(x,t)\,dx = (2/\Gamma(\beta+1))\,t^\alpha$, then the resulting process turns out to be self-similar with Hurst exponent $H = \alpha/2$ and the variance law is consistent with slow diffusion for $0 < \alpha < 1$ and fast diffusion for $1 < \alpha \le 2$. However it is worth noting to be remarked also that a linear variance growing is possible, but with non-Gaussian *pdf*, when $\beta \neq \alpha = 1$, and a Gaussian *pdf* with non-linear variance growing when $\beta = 1$ and $\alpha \neq 1$.

It is straightforward to note that, by using formula (13), evolution equation (8) reduces to the time-fractional diffusion if $\alpha = \beta < 1$, i.e.,

$$\frac{\partial P}{\partial t} = D_{RL}^{1-\beta}\frac{\partial^2 P}{\partial x^2}\,, \quad \text{with} \quad \mathcal{G}(x,t) = \frac{1}{2}\frac{1}{t^{\beta/2}}M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right)\,, \tag{25}$$

and variance $\langle x^2 \rangle = (2/\Gamma(\beta+1))\, t^{\beta}$, and, by using formula (14), it reduces to the stretched Gaussian diffusion if $\alpha \neq 1$ and $\beta = 1$, i.e.,

$$\frac{\partial P}{\partial t} = \alpha\, t^{\alpha-1} \frac{\partial^2 P}{\partial x^2}\,, \quad \text{with} \quad \mathcal{G}(x,t) \;=\; \frac{1}{2}\frac{1}{t^{\alpha/2}} M_{1/2}\left(\frac{|x|}{t^{\alpha/2}}\right) \;=\; \frac{1}{\sqrt{4\pi}}\frac{1}{t^{\alpha/2}} \exp\left\{-\frac{x^2}{4t^{\alpha}}\right\}\,, \tag{26}$$

and variance $\langle x^2 \rangle = 2\, t^{\alpha}$, and finally to the standard Gaussian diffusion if $\alpha = \beta = 1$, i.e., (1) with (2) and variance $\langle x^2 \rangle = 2\, t$. The Green functions of these last two cases, i.e., ($\alpha \neq 1$, $\beta = 1$) and ($\alpha = \beta = 1$), follows by (23) noting that $M_1(\tau) = \delta(\tau - 1)$.

In general, even if the Green functions are interpreted as one-point *pdf* evolving in time, they cannot determine a *unique* (self-similar) stochastic process because this requires the determination of any multi-point *pdf*. But, what concerns the ggBm, since the increments are stationary, it emerges to be uniquely determined by its covariance structure [21, 22]. Then, even if the ggBm is not Gaussian in general, it is a valuable example of a process defined only through its first and second moments, which indeed is a remarkable property of the Gaussian processes. Then the ggBm is a direct generalization of the Gaussian processes and, in the same way, the Mainardi function M_{ν} is a generalization of the Gaussian function, and it emerges to be the marginal *pdf* of non-Markovian diffusion processes that describe both slow and fast anomalous diffusion.

To conclude, in this Short Note it is highlighted *the relationship between the Erdélyi–Kober fractional operators and the valuable family of stochastic processes* generated by the ggBm, whose some remarkable properties are reported above, *and the key role of the Mainardi function in this framework*. In fact, the particle *pdf* of associated to the ggBm is the solution of a fractional integral equation (10), or analogously of a fractional diffusion equation (17), in the Erdélyi–Kober sense and this solution is a Mainardi function. Since the governing equation of these processes is a fractional equation in the Erdélyi–Kober sense it is proposed to called this family of diffusive processes as *Erdélyi–Kober fractional diffusion*.

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CRS4
Center for Advanced Studies, Research and Development in Sardinia
Polaris Bldg. 1
09010 Pula (CA) – ITALY
e-mail: pagnini@crs4.it

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