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The evolution equation for the radius of a premixed flame ball in fractional diffusive media

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Abstract. The evolution equation for the radius of an isolated premixed flame ball is derived in the framework of a new method that strongly simplifies previous ones and highlights that they are based on Gaussian modelling of diffusion. The main idea is to split the flame ball in two components: the inner kernel, which is driven by a Poisson-type equation with a general polynomial forcing term, and the outer part, which is driven by a generalized diffusion process valid for fractional diffusive media. The evolution equation for the radius of the flame ball is finally determined as the evolution equation for the interface that matches the solution of the inner spherical kernel and the solution of the outer diffusive part and it emerges to be a nonlinear fractional differential equation. The effects of fractional diffusion on stability of solution are also picked out.

1 Introduction

The topic of *Flame Balls* is an intriguing scientific issue beginning from its history. In fact, stable flame balls were theoretically predicted since 1944 by the Soviet physicist Ya. B. Zeldovich [1] but they were experimentally discovered only accidentally and much more recently in 1984, during short-duration drop tower experiments conducted by P. D. Ronney and collaborators [2,3]. They were finally experimentally established in 1998 from space flight experiment conducted on the STS-83/MSL-1 Space Shuttle mission [4] because a micro-gravity environment is needed to obtain spherical symmetry and to avoid buoyancy-induced extinction of the flame ball. See also `http://spaceresearch.nasa.gov/research_projects/sts-107_sofball.html`. Then experimentally based theoretical research is relatively recent.

A flame ball is an isolated three-dimensional combustion spot with spherical symmetry that occurs in a lean premixed mixture. Differently from the well-known classical nonpremixed combustion, where the fuel and the oxidant must be mixed before than combustion can take place, in the premixed combustion all reactants are intimately mixed at the molecular level before the combustion is started, and

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the combustion process is the one-step irreversible chemical reaction $Fresh\ Gas \rightarrow Burnt\ Gas + Heat$. Premixed combustion includes also the familiar laboratory Bunsen burner as well as the flame inside a gasoline-fueled internal combustion engine. Moreover, understanding combustion in lean conditions has a key role in product engineering because it is involved in designing of efficient, clean-burning combustion engines. In fact lean premixed combustion is characterized by low production of NO_x and particulate and then it is of paramount importance to challenge the environmental emergency and to meet future emission standard.

Evolution equations for the flame ball radius R have been derived [5–8] and they emerge to be nonlinear fractional differential equations of order 1/2. However all derivations are based on a complex matching of multiple asymptotic expansions. In the present paper a recent method proposed by the author [9,10] to derive the evolution equation for the flame ball radius is taken into account. It strongly simplifies previous methods and moreover it highlights that considered literature equations are founded on the classical diffusion process which is characterized by a Gaussian probability density function (*pdf*) and a linear growth of the variance of particle displacement, i.e $\langle x^2 \rangle \sim t$. Classical diffusion process is also referred to as normal diffusion, to distinguish it from anomalous diffusion in which the displacement variance grows nonlinearly in time, for example with the power law $\langle x^2 \rangle \sim t^\alpha$, with $\alpha > 0$. Generally, anomalous diffusion is met in complex media. Obviously the classical diffusion process is not more correct for describing anomalous diffusion processes [11]. Remembering the claim by Klafter and Sokolov that *Anomalous is normal* [11], in the framework of this recent method [9,10] the evolution equation for the flame ball radius is derived for fractional diffusive media.

In literature, anomalous diffusion is modeled in several different ways, however Fractional Calculus turns out to be one of the most successful tool [12]. Models based on fractional differential equations have been proposed in a large number of research fields. The main characteristic that relates fractional differential equations to anomalous diffusion is that, when the solution is interpreted as *pdf*, the particle displacement variance turns out to be driven by the fractional order of derivation. Here a general non-Markovian time fractional diffusion process is considered [13,14]. It can model both slow diffusion, i.e. $\langle x^2 \rangle \sim t^\alpha$ with $\alpha < 1$, and fast diffusion, i.e. $\langle x^2 \rangle \sim t^\alpha$ with $1 < \alpha \le 2$. Moreover, all the statistical moments of the *pdf* of particle displacement are finite, which is a remarkable property for physical applications and not met by some other fractional diffusion models, see e.g. [15]. It is important to remark and stress here that, previously, such a method has been just roughly outlined [9], without applications, and mathematically reconsidered [10] with sketched reference to the space-time fractional diffusion equation [15]. Here it is considered for the general non-Markovian time fractional diffusion process [13,14] and the effects of fractional diffusion on stability of solution are also picked out.

In Sec. 2 the scientific background is briefly reminded including Fractional Calculus, Zeldovich solution and literature equations for flame ball radius and fractional diffusion modelling. In Sec. 3 the new method is proposed and the equation for the evolution of the flame ball radius in fractional diffusive media is derived with a general polynomial forcing. In Sec. 4 the effect of anomalous diffusion on the stability of solution is highlighted and analyzed. Finally in Sec. 5 the conclusion and perspective for future developments are discussed.

2 Scientific background

2.1 Remind in fractional calculus

This introductory section to Fractional Calculus follows the 1996 CISM lectures by Gorenflo and Mainardi [16].

Let $f(t)$, with $t > 0$, be a sufficiently well-behaved function, Riemann–Liouville and Caputo fractional derivatives of real order $\mu > 0$ are both based on Riemann–Liouville fractional integral which is defined as

$$ {}_tJ^{\mu} f(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau)\, d\tau, \quad \mu > 0. \tag{1}$$

The operator ${}_tJ^{\mu}$ is conventionally the Identity operator when $\mu = 0$, i.e. ${}_tJ^0 = I$, and it meets the semi-group property

$$ {}_tJ^{\mu}\, {}_tJ^{\eta} = {}_tJ^{\eta}\, {}_tJ^{\mu} = {}_tJ^{\mu+\eta}, \quad \mu, \eta \geq 0. \tag{2}$$

The most simple and useful example of Riemann–Liouville fractional integration is the function $f(t) = t^{\nu}$, for $t > 0$,

$$ {}_tJ^{\mu}\, t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\mu)}\, t^{\nu+\mu}, \quad \mu \geq 0, \quad \nu > -1. \tag{3}$$

The Riemann–Liouville time fractional derivative of order $\mu > 0$ is defined, in analogy with the ordinary derivative, as the operator ${}_tD^{\mu}$ which is the left inverse of the Riemann–Liouville integral of order μ

$$ {}_tD^{\mu}\, {}_tJ^{\mu} = I, \quad \mu > 0. \tag{4}$$

If m denotes the positive integer such that $m - 1 < \mu \leq m$, then from (2) and (4) it follows that ${}_tD^{\mu} f(t) := {}_tD^{m}\, {}_tJ^{m-\mu} f(t)$. Hence for $m - 1 < \mu < m$

$$ {}_tD^{\mu} f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(\mathrm{m}-\mu)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\mu+1-m}}\, d\tau \right], \tag{5}$$

and ${}_tD^{\mu} f(t) = d^m f(t)/dt^m$ when $\mu = m$.

On the other hand, the fractional derivative of order $\mu > 0$ in the Caputo sense is defined as the operator ${}_tD_*^{\mu}$ such that ${}_tD_*^{\mu} f(t) := {}_tJ^{m-\mu}\, {}_tD^{m} f(t)$. Hence for $m-1 < \mu < m$

$$ {}_tD_*^{\mu} f(t) = \frac{1}{\Gamma(\mathrm{m}-\mu)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\mu+1-m}}\, d\tau, \tag{6}$$

and ${}_tD_*^{\mu} f(t) = d^m f(t)/dt^m$ when $\mu = m$. Thus, when the order is not integer the two fractional derivatives mainly differ because the derivative of order m does not generally commute with the fractional integral.

Furthermore, unlike Riemann–Liouville fractional derivative, Caputo fractional derivative of order μ, with $m - 1 < \mu \leq m$, satisfies the relevant property of being zero when it is applied to a constant, and, in general, when it is applied to any power function of non-negative integer degree less than m. Indeed, what concerns Riemann–Liouville derivative operator, for $t > 0$,

$$ {}_tD^{\mu} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\mu)} t^{\nu-\mu}, \quad \mu \geq 0, \quad \nu > -1. \tag{7}$$

Gorenflo and Mainardi [16] have shown the essential relationships between the two fractional derivatives (when both exist), for $m - 1 < \mu < m$, which are

$$ {}_tD^{\mu} f(t) = {}_tD^{\mu} g(t) \quad \text{if} \quad f(t) = g(t) + \sum_{j=1}^{m} k_j t^{\mu-j}, \tag{8}$$

$$ {}_tD_*^{\mu} f(t) = {}_tD_*^{\mu} g(t) \quad \text{if} \quad f(t) = g(t) + \sum_{j=1}^{m} k_j\, t^{m-j}, \tag{9}$$

where coefficients k_j are arbitrary constants, and

$$ {}_tD_*^{\mu} f(t) = {}_tD^{\mu} \left[f(t) - \sum_{n=0}^{m-1} f^{(n)}(0^+) \frac{t^n}{n!} \right], \tag{10} $$

that applying (7) becomes

$$ {}_tD_*^{\mu} f(t) = {}_tD^{\mu} f(t) - \sum_{n=0}^{m-1} \frac{f^{(n)}(0^+) t^{n-\mu}}{\Gamma(\mathrm{n} - \mu + 1)}. \tag{11} $$

Caputo fractional derivative is a regularization in the time origin of the Riemann–Liouville fractional derivative. From (11) it emerges that Caputo fractional derivative exists if all the limiting values $f^{(n)}(0^+) := \lim_{t\to 0^+} f^{(n)}(t)$ are finite for $n = 0, 1, 2, \ldots, m - 1$. Further on the basic theory of Fractional Calculus can be found elsewhere in this Issue, in particular, for what concerns the sufficient and necessary conditions of the existence of the Riemann–Liouville and Caputo fractional derivation, please see [17].

2.2 Zeldovich flame ball and its radius evolution equation in Gaussian diffusive media

Stable flame ball was theoretically predicted in 1944 by the Soviet physicist Ya. B. Zeldovich [1] as exact solution to the heat and mass conservation equations in spherical geometry with radial coordinate denoted by r,

$$ \rho C_p \left(\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial r} \right) = h \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + QW, \tag{12a} $$

$$ \rho \left(\frac{\partial Y_F}{\partial t} + U \frac{\partial Y_F}{\partial r} \right) = \rho D_F \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) - W, \tag{12b} $$

where T is the temperature, Y_F the mass fraction of the fresh gas, U the radial velocity, W the chemical rate, Q the heat of reaction, ρ the specific mass, C_p the specific heat at constant pressure, h the heat conductivity, D_F the diffusion coefficient of the fresh gas. Temperature and mass concentration fields are related in (12) by

$$ Le(T - T_\infty) = \frac{Q}{C_p} \left(1 - \frac{Y_F}{Y_\infty} \right), \tag{13} $$

where the nondimensional number $Le = h/(\rho\ D_F\ C_p)$ is called Lewis number and T_∞ and Y_∞ are the reference values for temperature and mass fraction of fresh gas, respectively.

After transformation (13), and setting without loss of generality $T_\infty = 0$ and $Y_\infty = 1$, equation for T (12a) gives

$$ \rho \left(\frac{\partial Y_F}{\partial t} + U \frac{\partial Y_F}{\partial r} \right) = Le \rho D_F \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) - LeW. \tag{14} $$

When, for large activation energy, the chemical source term behaves like a Dirac δ-function at the flame sheet [6], the solutions to steady, convection-free diffusion equations for temperature and chemical species concentration, i.e.

$$ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = 0 \quad \text{and} \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial Y_F}{\partial r} \right) = 0, \tag{15} $$

are of the form $c_1 + c_2/r$, where c_1 and c_2 are constants, as it has been clearly reviewed by Ronney [4]. This form satisfies the requirement that T and Y_F be bounded as $r \to \infty$. For cylindrical and planar geometry the corresponding forms are $c_1 + c_2 \ln r$ and $c_1 + c_2 r$, respectively, which are obviously unbounded as $r \to \infty$. For this reason theory admits steady flame ball solutions, but not steady "flame cylinder" or steady "flame slab" solutions.

Zeldovich showed that, for an adiabatic flame ball, the temperature at the surface of the flame ball T_* is $T_* = T_{ad}/Le$, where T_{ad} is the adiabatic homogeneous flame temperature. Then, inside the ball ($r < R$), the temperature profile $T(r)$ is constant and corresponds to the burnt gas temperature and, outside the ball ($r > R$), it decreases depending on the flame ball radius R with the law

$$T(r) = T_* \frac{R}{r}, \quad r > R. \tag{16}$$

What concerns the fresh gas mass fraction, it is null inside the ball and, as it follows from formula (13), it increases outside the ball as

$$Y_F = 1 - \frac{R}{r}, \quad r > R, \tag{17}$$

where $C_p T_{ad}/Q = 1$. Flame balls can exist if $T_* > T_{ad}$ and this condition is met when $Le < 1$, while conventional propagating flames are observed under any value of Lewis number. The reason is that for $T_* < T_{ad}$, $Le > 1$, the flame balls are weaker than plane flames.

This steady state can be realized only if the flame ball radius R is constant in time. Then, the evolution equation for the flame ball radius and the later analysis on the stability of the solution are necessary. In literature, such equation is derived by matching multiple asymptotic expansions. The first valuable result was obtained in 1985 by Joulin [5] neglecting heat losses. Joulin seminal equation is

$$R(t)_t D_*^{1/2} R(t) = R(t) \ln R(t) + Eq(t), \tag{18}$$

where $Eq(t)$ is a measure of the energy input with intensity $E > 0$ and temporal dependence as $q(t)$. Later heat losses were included and in 1990 Buckmaster, Joulin and Ronney [6,7] derived the following equation

$$R(t)_t D_*^{1/2} R(t) = R(t) \ln R(t) + Eq(t) - \lambda R^3(t), \tag{19}$$

where λ is associated to heat losses. More recently, on the basis of the linearized Eddington equation for radiative field, Guyonne and Noble [8] derived in 2007 the following evolution equation for the radius of a flame ball

$$R(t)_t D_*^{1/2} R(t) = R(t) \ln R(t) + Eq(t) - 3\lambda R^2(t). \tag{20}$$

The above nonlinear equations (18), (19), (20) are very difficult to be managed, however successful numerical schemes have been developed [8,18–21]. In §3, the recent simple method of derivation [9,10] is re-presented because it turns out to be useful to highlight and clarify which are the main aspects that constitute and drive the process and then to help the advance of research on this topic, in particular on finding analytical and/or numerical solution, analyzing solution properties and study solution stability.

2.3 Fractional diffusion modelling

Normal or Gaussian diffusion is a process driven by the Markovian classical diffusion equation

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2}, \quad x \in \mathcal{R}, \quad t \in \mathcal{R}_0^+, \tag{21}$$

with $P(x, 0) = P_0(x)$, where $P(x, t)$ is the *pdf* to find a particle in x at time t. The fundamental solution of (21) corresponding to $P_0(x) = \delta(x)$, named also Green function, is the Gaussian density

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{-\frac{x^2}{4t}\right\}, \tag{22}$$

whose variance grows linearly in time, i.e. $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x,t)\, dx = 2\, t$.

In order to generalize the classical Markovian setting to Non-Markovian cases, the following non-Markovian diffusion equation has been introduced by Mura, Taqqu and Mainardi [13]

$$P(x,t) = P_0(x) + \int_0^t \frac{\partial g(s)}{\partial s} K[g(t) - g(s)] \frac{\partial^2 P(x,s)}{\partial x^2}\, ds, \quad x \in \mathcal{R}, \quad t \in \mathcal{R}_0^+, \tag{23}$$

where $K(t)$ is a memory kernel and $g(t)$, with $g(0) = 0$, is a smooth and increasing function describing a time stretching. The Green function of equation (23) turns out to be

$$\mathcal{G}(x,t) = \int_0^\infty f(x,\tau) h(\tau, g(t))\, d\tau, \tag{24}$$

where $f(x, t)$ is the Gaussian solution (22) of the Markovian diffusion process, i.e. $K(t) = \delta(t)$, and $h(\tau, t)$ is the fundamental solution of the so-called *non-Markovian forward drift equation*

$$u(\tau,t) = u(\tau,0) - \int_0^t K(t-s) \frac{\partial u(\tau,s)}{\partial \tau}\, ds, \quad \tau, t \in \mathcal{R}_0^+. \tag{25}$$

When the kernel and the time-stretching functions are determined as

$$K(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad g(t) = t^{\alpha/\beta}, \quad 0 < \alpha \le 2, \quad 0 < \beta \le 1, \tag{6.5}$$

equation (23) becomes [14,22]

$$P(x,t) = P_0(x) + \frac{1}{\Gamma(\beta)} \frac{\alpha}{\beta} \int_0^t \tau^{\alpha/\beta - 1} (t^{\alpha/\beta} - \tau^{\alpha/\beta})^{\beta-1} \frac{\partial^2 P(x,\tau)}{\partial x^2}\, d\tau, \tag{26}$$

that in diffusion form reads

$$\frac{\partial P}{\partial t} = \frac{\alpha}{\beta} t^{\alpha/\beta - 1} \left[{}_{t^{\alpha/\beta}} D^{1-\beta} \frac{\partial^2 P}{\partial x^2} \right]. \tag{27}$$

The Green function of equation (27) is [14,22]

$$\mathcal{G}(x,t) = \frac{1}{2} \frac{1}{t^{\alpha/2}} M_{\beta/2}\left(\frac{|x|}{t^{\alpha/2}}\right), \tag{28}$$

where $M_\nu(z)$ is the M-Wright function [23], also named Mainardi function see e.g. [24], which is defined as

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma[-\nu \mathrm{n} + (1-\nu)]} = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\Gamma(\nu)\sin(\pi\nu \mathrm{n}), \quad 0<\nu<1\,, \tag{29}$$

see Ref. [25] for a review, here it is reminded only the special case $M_{1/2}(z) = (1/\sqrt{\pi})\exp(-z^2/4)$.

It is worth noting to remark here that the non-Markovian fractional kinetic Eq. (26) can be expressed in terms of the Erdélyi–Kober fractional integral $I_\eta^{\gamma,\delta}$ as follows

$$P(x,t) = P_0(x) + t^\alpha \left[I_{\alpha/\beta}^{0,\beta} \frac{\partial^2 P}{\partial x^2}\right], \tag{30}$$

where the Erdélyi–Kober fractional integral operator $I_\eta^{\gamma,\delta}$ of a function $v(t)$ is defined as [26, formula (1.1.17)]

$$I_\eta^{\gamma,\delta}\, v(t) = \frac{\eta}{\Gamma(\delta)} t^{-\eta(\delta+\gamma)} \int_0^t \tau^{\eta(\gamma+1)-1}(t^\eta - \tau^\eta)^{\delta-1} v(\tau)\, d\tau. \tag{31}$$

The variance of (28) is $\langle x^2\rangle = \int_{-\infty}^{+\infty} x^2 \mathcal{G}(x,t)\, dx = (2/\Gamma(\beta+1))\, \mathrm{t}^\alpha$, then the resulting process turns out to be self-similar with Hurst exponent $H = \alpha/2$ and the variance law is consistent with slow diffusion for $0<\alpha<1$ and fast diffusion for $1<\alpha\le 2$. However it is worth noting to be remarked also that a linear variance growing is possible, but with non-Gaussian *pdf*, when $\beta \ne \alpha = 1$. It is straightforward to note that the evolution equation (26) reduces to time-fractional diffusion if $\alpha = \beta < 1$, i.e.

$$\frac{\partial P}{\partial t} = {}_tD^{1-\beta}\frac{\partial^2 P}{\partial x^2}, \quad \text{with} \quad \mathcal{G}(x,t) = \frac{1}{2}\frac{1}{t^{\beta/2}} M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right), \tag{32}$$

and variance $\langle x^2\rangle = (2/\mathit{\Gamma}(\beta+1))\, t^\beta$, to stretched Gaussian diffusion if $\alpha \ne 1$ and $\beta = 1$, i.e.

$$\frac{\partial P}{\partial t} = \alpha t^\alpha \frac{\partial^2 P}{\partial x^2}, \quad \text{with} \quad \mathcal{G}(x,t) = \frac{1}{2}\frac{1}{t^{\alpha/2}} M_{1/2}\left(\frac{|x|}{t^{\alpha/2}}\right) = \frac{1}{\sqrt{4\pi}}\frac{1}{t^{\alpha/2}}\exp\left\{-\frac{x^2}{4t^\alpha}\right\}, \tag{33}$$

and variance $\langle x^2\rangle = 2t^\alpha$, and finally to standard Gaussian diffusion if $\alpha = \beta = 1$, i.e. (21) with (22) and variance $\langle x^2\rangle = 2\,t$.

The M-function emerges to be the marginal *pdf* of non-Markovian diffusion processes that describe both slow and fast anomalous diffusion. A detailed stochastic analysis of above non-Markovian modelling of anomalous diffusion is performed in [13,14,25,27], it follows that the M-Wright function $M_\nu(z)$ has for anomalous diffusion the same key role of the Gaussian density for standard and fractional Brownian motions. Here only fractional processes with finite statistical moments are considered then processes which involves space fractional derivative operators are not taken into account because infinite moments can occur.

To conclude, different phenomena are described by different fractional differential equations which could be also nonlinear, see e.g. [28]. However, in all cases, if the process is self-similar and the variance is proportional to t^α, i.e. $\langle x^2\rangle \sim t^\alpha$ with $\alpha > 0$, then the Green function has the general form

$$\mathcal{G}(x,t) = \frac{1}{t^{\alpha/2}}\mathcal{H}\left(\frac{x}{t^{\alpha/2}}\right). \tag{34}$$

In fact

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \mathcal{G}(x,t)\, dx = \int_{-\infty}^{+\infty} \frac{x^2}{t^{\alpha/2}} \mathcal{H}\left(\frac{x}{t^{\alpha/2}}\right) dx$$
$$= \left\{ \int_{-\infty}^{+\infty} \eta^2 \mathcal{H}(\eta)\, d\eta \right\} \cdot t^{\alpha} = const \cdot t^{\alpha}, \qquad (35)$$

after application of the change of variable $x = \eta t^{\alpha/2}$ and provided that $\int_{-\infty}^{+\infty} \eta^2 \mathcal{H}(\eta) d\eta < \infty$.

3 The flame ball radius evolution equation in fractional diffusive media

3.1 Description of the method

Let R be at any fixed instant t the radius of the flame ball, then its growing in time is here assumed to be determined by the evolution of the matching interface between an inner kernel ($r < R$), which is the quasi-stationary spherical solution of a Poisson-type equation, and an outer diffusive part ($r > R$), which is the solution of a diffusion equation.

Let Φ_{s} be the inner solution and Φ_{d} be the outer solution. Then the growing in time of the flame ball radius is determined by a diffusion operator that acts on the inner solution computed on the surface of the flame ball. This means that the source term of the diffusion process is determined by $\Phi_{\mathrm{s}}(\mathrm{x},\mathrm{t})\delta(\mathrm{x}-\mathrm{R}(\mathrm{t}))$ and the action of the operator emerges to be a double convolution integral both in space and time with propagating kernel $\mathcal{K}(x,t)$, i.e.

$$R(t) = \mathcal{K}(x,t) * \Phi_{\mathrm{s}}(\mathrm{x},\mathrm{t})\delta(\mathrm{x}-\mathrm{R}(\mathrm{t})) = \Phi_{\mathrm{d}}(\mathrm{R},\mathrm{t}). \qquad (36)$$

This method has been recently proposed by the author [9,10]. It has been suggested by the diffusive formulation discussed in [18,29,30]. Moreover, such diffusive formulation, has been used by Gorenflo and Vessella [31] to study Volterra integral equations.

3.2 The inner solution

Consider a flame initiated by a point source energy input, which spherically evolves under the action of a radial forcing $\sim 1/r^2$, radiative heat losses and any possible forcing. Then the inner solution in spherical coordinates $\Phi_{\mathrm{s}}(\mathrm{r},\mathrm{t})$ is determined as the quasi-stationary solution of the Poisson-type equation with a general polynomial forcing

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \Phi_{\mathrm{s}}}{\partial r}\right] = \frac{2}{r^2} - 2\lambda \sum_{i=1}^{n} 6\gamma_i r^{\xi_i}, \qquad (37)$$

with the boundary condition

$$\left[r^2 \frac{\partial \Phi_{\mathrm{s}}}{\partial r}\right]_{r=0} = -2\, Eq(t), \quad q(0) = 0, \qquad (38)$$

where $Eq(t)$ is the energy input with intensity $E > 0$ and temporal variation $q(t)$. The numerical factors on RHS of (37) and (38) are chosen for formal reasons. Finally, the inner solution $\Phi_{\mathrm{s}}(\mathrm{r},\mathrm{t})$ turns out to be

$$\Phi_{\mathrm{s}}(\mathrm{r},\mathrm{t}) = 2\left[\ln \mathrm{r} + \frac{\mathrm{E\,q(t)}}{\mathrm{r}} - \lambda \mathrm{r}^2 \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{6\gamma_{\mathrm{i}} \mathrm{r}^{\xi_{\mathrm{i}}}}{(\xi_{\mathrm{i}}+3)(\xi_{\mathrm{i}}+2)}\right] = 2\,\mathrm{F}(\mathrm{r},\mathrm{t}). \qquad (39)$$

3.3 The outer solution

Each point of the matching interface is assumed to be diffused along the one-dimensional axes that ranges from $-\infty$ to $+\infty$ and is aligned with r. Then, the spherical reference system characterized by $r > 0$, which was used to determine the growing of the inner solution Φ_s, is now abandoned to use a one-dimensional Cartesian axes x, such that $|x| = r$, and the diffusion is modelled with respect to this reference frame. This means that now the flame ball radius is located in $|x| = R$.

Finally, the outer diffusive solution Φ_d is determined as the solution of a diffusion equation with source term S given by the inner solution computed in the inner-outer interface located at the flame position $R(t)$. In the r-coordinate system $S(r,t) = \Phi_s(\mathrm{r,t})\delta(\mathrm{r-R(t)}) = 2\mathrm{F(r,t)}\delta(\mathrm{r-R(t)})$ and in the x-coordinate system $S(x,t) = \Phi_s(\mathrm{x,t})\delta(\mathrm{x-R(t)}) = 2\mathrm{F(x,t)}\delta(\mathrm{x-R(t)})$.

Anomalous diffusion is characterized by a nonlinear growing rate in time of the variance, here the following power law is considered: $\langle x^2\rangle \sim t^\alpha$, $0 < \alpha \le 2$. In §2.2 it has been pointed out that different types of anomalous diffusion equation have been proposed in literature. However, from all evolution equations to model anomalous diffusion which admit a self-similar solution, Green function emerges to be expressed by formula (34). Then the solution of the whole diffusion process with source term $S(x,t) = \Phi_s(\mathrm{x,t})\delta(\mathrm{x-R(t)}) = 2\mathrm{F(x,t)}\delta(\mathrm{x-R(t)})$ is given by the double convolution integral

$$\Phi_d(\mathrm{x,t}) = 2\int_{-\infty}^{+\infty}\int_0^{\mathrm{t}} \mathcal{G}(\mathrm{x}-\eta,\mathrm{t}-\tau)\mathrm{F}(\eta,\tau)\delta(\eta-\mathrm{R}(\tau))\,d\eta\,d\tau, \tag{40}$$

which after computing the convolution in space reduces to

$$\Phi_d(\mathrm{x,t}) = 2\int_0^{\mathrm{t}} \mathcal{G}(\mathrm{x}-\mathrm{R}(\tau),\mathrm{t}-\tau)\mathrm{F}(\mathrm{R}(\tau),\tau)\,d\tau. \tag{41}$$

To conclude, inserting (34) in the above formula, the generalized outer solution turns out to be

$$\Phi_d(\mathrm{x,t}) = 2\int_0^{\mathrm{t}} \mathcal{H}\left[\frac{\mathrm{x}-\mathrm{R(t)}}{(\mathrm{t}-\tau)^{\alpha/2}}\right]\frac{\mathrm{F}(\mathrm{R}(\tau),\tau)}{(\mathrm{t}-\tau)^{\alpha/2}}\,d\tau. \tag{42}$$

3.4 The evolution equation

Comparing (36) and (42) it emerges that the propagator $\mathcal{K}(x,t)$ is the Green function (34) and the evolution equation for the flame ball radius follows to be

$$R(t) = 2\mathcal{H}(0)\int_0^t \frac{F(R(\tau),\tau)}{(t-\tau)^{\alpha/2}}\,d\tau = \mathcal{N}\,{}_tJ^{1-\alpha/2}[F(R(t),t)], \tag{43}$$

with initial condition $R(0) = 0$, where $\mathcal{N} = 2\mathcal{H}(0)\Gamma(1-\alpha/2)$ and ${}_tJ^{1-\alpha/2}$ is the Riemann–Liouville fractional integral of order $1-\alpha/2$ defined in (1). Applying the Riemann–Liouville time-fractional derivative operator ${}_tD^{1-\alpha/2}$, which is defined in (5), on both sides of (43) gives

$${}_tD^{1-\alpha/2}R(t) = \mathcal{N}\,{}_tD^{1-\alpha/2}\,{}_tJ^{1-\alpha/2}[F(R(t),t)] = \mathcal{N}\,F(R(t),t), \tag{44}$$

where property (4) is used. After multiplication by $R(t)$, the evolution equation (44) becomes the following nonlinear fractional differential equation

$$R(t)D_t^{1-\alpha/2}R(t) = \mathcal{N}\left[R(t)\ln R(t) + Eq(t) - \lambda R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i R^{\xi_i}(t)}{(\xi_i+3)(\xi_i+2)}\right]. \tag{45}$$

Relationship (11) between Riemann–Liouville ${}_tD^{\mu}$ and Caputo ${}_tD_*^{\mu}$ fractional derivatives can be applied in (45). Since the order of fractional derivation is $0 < 1-\alpha/2 < 1$ and $R(0^+) = 0$, then ${}_tD^{1-\alpha/2}R(t) = {}_tD_*^{1-\alpha/2}R(t)$. Finally, in terms of Caputo time-fractional derivative, the evolution equation of the flame radius is

$$R(t){}_tD_*^{1-\alpha/2}R(t) = \mathcal{N}\left[R(t)\ln R(t) + Eq(t) - \lambda R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i R^{\xi_i}(t)}{(\xi_i+3)(\xi_i+2)}\right], \quad (46)$$

or analogously

$$R(t){}_tD_*^{1-\alpha/2}R(t) = R(t)\ln R^{\mathcal{N}}(t) + \mathcal{N}Eq(t) - \lambda\mathcal{N}R^3(t)\sum_{i=1}^{n}\frac{6\gamma_i R^{\xi_i}(t)}{(\xi_i+3)(\xi_i+2)}. \quad (47)$$

For normal diffusive media the Green function (34) corresponds to the Gaussian density (22) and $\alpha = 1$. Then, noting that $\mathcal{H}(0) = 1/(2\sqrt{\pi})$ and remembering that $\Gamma(1/2) = \sqrt{\pi}$, the factor $\mathcal{N} = 2\mathcal{H}(0)\Gamma(1-1/2) = 1$. Moreover for $n = 1$, if $\gamma_1 = 1$ and $\xi_1 = 0$ the generalized evolution equation (46) reduces to Buckmaster–Joulin–Ronney equation (19), as well as to Joulin equation (18) setting also $\lambda = 0$; and if $\gamma_1 = 1$ and $\xi_1 = -1$ it reduces to Guyonne–Noble equation (20).

4 Solution stability and Green function of diffusion

The problem of stability of the flame ball is important to theoretically design the experimental realization of the stable flame balls predicted by Zeldovich, but also for applicative reasons, to maintain the combustion in the most efficient regime and to prevent the quenching of the flame, and for security reason, to avoid that the radius diverges.

Literature analysis on solution stability is performed for Joulin equation (18) and Buckmaster–Joulin–Ronney equation (19) where Gaussian diffusion is assumed so that $\mathcal{N} = 1$. For details on stability of solution of (19), the interest reader is referred to [30,32,33] and to [5,29,34] for the analysis of solution of the original Joulin equation (18) without heat losses. Here it is briefly reminded by literature that when radiative heat losses are larger than a critical value, i.e. $\lambda > \lambda_{cr}$, then the flame always quenches; otherwise when $\lambda < \lambda_{cr}$ the flame quenches if $E < E_{cr}(q)$ and it stabilizes to R_2 (or R_1) if $E > E_{cr}(q)$ (or $E = E_{cr}(q)$), where $R_2 > R_1$ are the solutions of the equation $\ln R = \lambda R^2$. Stability and threshold phenomenon are analyzed for the general case with a polynomial forcing but omitting the logarithmic function in [32, §5].

However the quantity $\mathcal{N} = 2\mathcal{H}(0)\Gamma(1-\alpha/2)$ is much more then a neutral multiplicative factor as follows from (47). This means that the stability analysis for equations in fractional diffusive media when $\mathcal{N} \neq 1$ gives different results from that for normal diffusive media when $\mathcal{N} = 1$, at least quantitatively on the determination of the threshold value. In particular, from the definition of $\mathcal{N}$ it emerges that this difference is due to $\mathcal{H}(0)$ and then to the Green function $\mathcal{G}(x,t)$ when $x = 0$, but this means also that the whole diffusive process is of paramount importance because the behaviour of $\mathcal{G}(0,t)$ is peculiar of each process. Different behaviours of $\mathcal{G}(0,t)$ for the cases listed in Table 1 are plotted in Fig. 1. What emerges from the plots is that in slow diffusion cases *a*), *d*), *e*) with $\alpha = 0.5$, the decreasing of $\mathcal{G}(0,t)$ is quicker for short elapsed times ($t < 1$) and slower for large times ($t > 1$) than the normal case *g*), in particular closer is β to the normal diffusion value $\beta = 1$ then closer is $\mathcal{G}(0,t)$ to the normal case *g*). What concerns fast diffusion cases *c*), *f*) with $\alpha = 1.5$, the behaviour in time with respect to the normal case *g*) is the opposite of slow diffusion

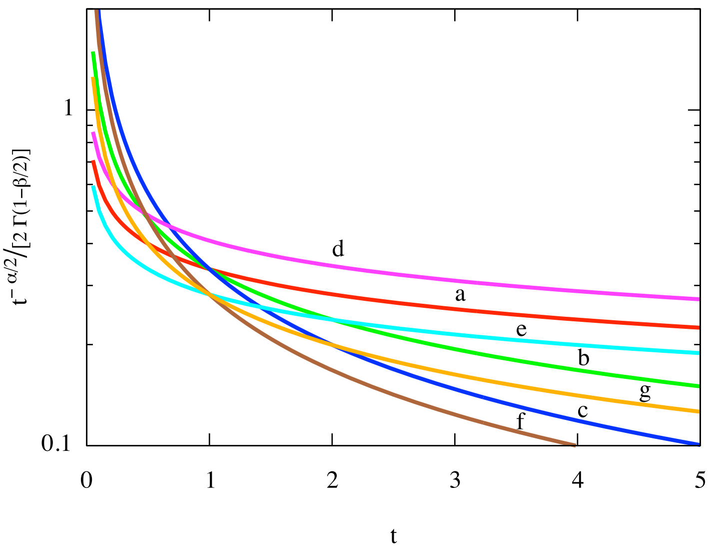

Fig. 1. Plots of the Green function value in the origin $x = 0$, i.e. $\mathcal{G}(0, t) = t^{-\alpha/2}/[2\Gamma(1 - \beta/2)]$, for different pairs of parameters $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. Please see Table 1 for the meaning of labels.

Table 1. Label list for Figure 1. The numbers in brackets refer to the corresponding formula of Green function $\mathcal{G}(x, t)$ along the text.

	$0 < \alpha \leq 2$ $0 < \beta < 1$	$0 < \alpha = \beta < 1$	$0 < \alpha \leq 2$ $\alpha \neq \beta = 1$	$\alpha = \beta = 1$
$\mathcal{G}(x, t)$	(28)	(32)	(33)	(22)
slow	**a)** $\alpha = 0.5; \beta = 0.8$	**d)** $\alpha = \beta = 0.5$	**e)** $\alpha = 0.5; \beta = 1$	
normal	**b)** $\alpha = 1\ \ ; \beta = 0.8$			**g)** $\alpha = \beta = 1$
fast	**c)** $\alpha = 1.5; \beta = 0.8$		**f)** $\alpha = 1.5; \beta = 1$	

but also in these cases closer is β to 1 closer is $\mathcal{G}(0, t)$ to case g). Finally the second linear case b) is very close to g) at the initial instants and it separates when time increases but since in this case $\beta < 1$ the behaviour of $\mathcal{G}(0, t)$ is of the same type of slow diffusion.

However, noting from (29) that $M_\nu(0) = 1/\Gamma(1-\nu)$, Green function (28) in $x = 0$ is $\mathcal{G}(0, t) = t^{-\alpha/2}/[2\Gamma(1 - \beta/2)]$, so that $\mathcal{H}(0) = 1/[2\Gamma(1 - \beta/2)]$. This means that $\mathcal{N} = \Gamma(1 - \alpha/2)/\Gamma(1 - \beta/2)$. Then the special case $\mathcal{N} = 1$ is recovered also for anomalous diffusion processes with $\alpha = \beta$. To conclude, the estimation of the correct diffusive process and then of the correct Green function $\mathcal{G}(x, t)$ has an important role on the determination of constant $\mathcal{N}$ which results on the nonlinearity of the equation, see (47). These differences on nonlinearity are not only due to the slow, fast or linear nature of the anomalous diffusion process, as intuitively one aspects, but also to the ratio α/β and have influence on the stability of the solution. A detailed stability analysis for solution of (46) can be performed, as a further development of the present work, by adopting the same numerical schemes discussed for equations (18), (19), (20) [8, 18–21].

5 Conclusion

In the present paper the problem of the derivation of the evolution equation for the flame ball radius is addressed and a recent method [9,10] is discussed. A general non-Markovian time fractional diffusion process [13,14] is considered and the effects of fractional diffusion on stability of solution are also picked out. This new method is based on the idea to split the flame ball in two components: the inner kernel, which is driven by a Poisson-type equation with a general polynomial forcing term, and the outer part, which is driven by a generalized anomalous diffusion process that holds for diffusion in fractional diffusive media. The evolution equation for the radius of the flame ball is determined as the evolution equation for the interface that matches the solution of the inner spherical kernel and the solution of the outer diffusive part. This method highlights that previous methods were based on classical Gaussian diffusion. The resulting equation turns out to be a nonlinear fractional differential equation whose fractional order of derivation emerges to be related to the diffusion process. In fact, the exponent of the power law of displacement variance $\langle x^2 \rangle \sim t^\alpha$ drives the order of fractional derivation which turns out to be $1 - \alpha/2$ and it reduces to $1/2$ when the diffusion process is Gaussian, i.e. $\alpha = 1$.

What concerns the stability of the solution it is emerged that it is strongly influenced by the diffusion process, in particular by the law of the variance growing and the shape of the Green function. Such influence is displayed by differences on nonlinearity.

This method strongly simplifies and generalizes previous derivations. In fact since a polynomial forcing and anomalous diffusion are considered, literature equations (18), (19), (20) [5–8] are recovered when the forcing and the diffusion process are appropriately chosen.

The main remarkable aspect of this new method is that, due to its clear and simple derivation, it can be a useful tool to further development and advance in the research on this topic helping to overcame the difficulties that the current models meet. In fact, the mathematical simplicity of equation foundation can highlight new promising way to find analytical and numerical solution, solution properties as well as to analyse solution stability which is of paramount importance for establishing the experimental configuration to observe the steady flame ball originally predicted by Zeldovich.

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References

1. Ya.B. Zeldovich, *Theory of Combustion and Detonation of Gases* (USSR Academy of Sciences, Moscow, 1944)
2. P.D. Ronney, Combust. Flame **82**, 1 (1990)
3. P.D. Ronney, K.N. Whaling, A. Abbud-Madrid, J.L. Gatto, V.L. Pisowicz, AIAA J. **32**, 569 (1994)
4. P.D. Ronney, M.S. Wu, H.G. Pearlman, K.J. Weiland, AIAA J. **36**, 1361 (1998)
5. G. Joulin, Combust. Sci. Tech. **185**, 99 (1985)
6. J.D. Buckmaster, G. Joulin, P.D. Ronney, Combust. Flame **79**, 381 (1990)
7. J.D. Buckmaster, G. Joulin, P.D. Ronney, Combust. Flame **84**, 411 (1991)
8. V. Guyonne, P. Noble, J. SIAM Appl. Math. **67**, 854 (2007)

9. G. Pagnini, in *Proceedings of FDA10. The 4th IFAC Workshop Fractional Differentiation and its Applications*, edited by I. Podlubny, B.M. Vinagre Jara, YQ. Chen, V. Feliu Batlle, I. Tejado Balsera (ISBN 9788055304878, 2010), Article no. FDA10-063
10. G. Pagnini, Fract. Calc. Appl. Anal. **14**, 80 (2011)
11. J. Klafter, I.M. Sokolov, Physics World **August**, 29 (2005)
12. I.M. Sokolov, J. Klafter, A. Blumen, Physics Today **November**, 48 (2002)
13. A. Mura, M.S. Taqqu, F. Mainardi, Physica A **387**, 5033 (2008) `http://arxiv.org/abs/0712.0240`
14. A. Mura, G. Pagnini, J. Phys. A **41**, 285003 (2008) `http://arxiv.org/abs/0801.4879`
15. F. Mainardi, G. Pagnini, Fract. Calc. Appl. Anal. **4**, 153 (2011)
16. R. Gorenflo, F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, edited by A. Carpinteri, F. Mainardi (Springer-Verlag, Wien and New York, 1997), p. 223, `http://arxiv.org/abs/0805.3823`
17. C.P. Li, Z.G. Zhao, Eur. Phys. J. Special Topics **193**, 5 (2011)
18. J. Audounet, J.-M. Roquejoffre, in *ESAIM: Proceedings Fractional Differential Systems: Models, Methods and Applications*, edited by D. Matignon G. Montseny, Vol. 5 (SMAI, Paris, 1998), p. 15, `http://www.emath.fr/proc/vol.5/`
19. J. Audounet, J.-M. Roquejoffre, H. Rouzaud, Math. Modelling Numer. Anal. **36**, 273 (2002)
20. F. Dubois, S. Mengué, Numerical Algorithms **34**, 303 (2003)
21. K. Diethelm, M. Weilbeer, Fract. Calc. Appl. Anal. **7**, 1 (2004)
22. A. Mura, Ph.D. thesis, University of Bologna, 2008 `http://www.fracalmo.org/Mura/`
23. F. Mainardi, Chaos, Solitons Fractals **7**, 1461 (1996)
24. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999)
25. F. Mainardi, A. Mura, G. Pagnini, Int. J. Diff. Equations **2010**, 104505 (2010) `http://www.hindawi.com/journals/ijde/2010/104505.html`
26. V. Kiryakova, *Generalized Fractional Calculus and Applications* (Longman, Harlow, 1994)
27. A. Mura, Mainardi F., Integral Transforms Spec. Funct. **20**, 185 (2009) `http://arxiv.org/abs/0711.0665`
28. E.K. Lenzi, M.K. Lenzi, L.R. Evangelista, L.C. Malacarne, R.S. Mendes, J. Stat. Mech., P02048 (2009)
29. J. Audounet, V. Giovangigli, J.-M. Roquejoffre, Physica D **121**, 295 (1998)
30. H. Rouzaud, C.R. Acad. Sci. Paris Série I **332**, 1083 (2001)
31. R. Gorenflo, S. Vessella, *Abel Integral Equations. Analysis and Applications* (Springer–Verlag, Berlin, 1991)
32. H. Rouzaud, Revista Matem. Compl. **16**, 207 (2003)
33. J.-M. Roquejoffre, H. Rouzaud, J. Comput. Appl. Math. **190**, 376 (2006)
34. C. Lederman, J.-M. Roquejoffre, N. Wolanski, C.R. Acad. Sci. Paris Série I **334**, 569 (2002)