Influence of Eulerian and Lagrangian scales on the relative dispersion properties in Lagrangian stochastic models of turbulence

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The influence of Eulerian and Lagrangian scales on the turbulent relative dispersion is investigated through a three-dimensional Eulerian consistent Lagrangian stochastic model. As a general property of this class of models, it is found to depend solely on a parameter β based on the Kolmogorov constants C_K and C_0. This parameter represents the ratio between the Lagrangian and Eulerian scales and is related to the intrinsic inhomogeneity of the relative dispersion process. In particular, the quantity $g^* = g/C_0$ (where g is the Richardson constant) and the temporal extension of the t^3 regime are found to be strongly dependent on the value adopted for β.

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In the frame of the Eulerian description of turbulence, the Kolmogorov [1,2] theories are based on spatial separation, while the natural description in the Lagrangian approach is in terms of time elapsing from an initial condition (see, e.g., Ref. [3]). Each description naturally leads to the definition of two characteristic scales, say λ and τ, for space and time, respectively. The effects of the two scales on turbulence dynamics are particularly evident in the relative dispersion process. When the particle separation lies in the inertial subrange, the spatial structure (Eulerian length scale) affects the dispersion features (Lagrangian property) [4]. The relative dispersion in the inertial subrange regime is characterized by the Richardson law [5] and the nondimensional constant g which should be considered universal even though measured values range from 0.06 to 6 [6–9].

Lagrangian stochastic models (LSMs), along with their utility in dispersion studies, provide a powerful tool for investigating some properties of turbulence, as shown in Refs. [10–13]. Its formulation naturally connects Lagrangian and Eulerian statistics through the requirement of statistical Eulerian consistency (well mixed condition, hereinafter WMC, [14,15]) of Lagrangian particle trajectories. Accordingly, model results should be dependent on the values of λ and τ. The aim of this Brief Report is to investigate the properties of WMC formulation of LSMs with respect to the parameters determining changes in the duration and the value of g of the Richardson t^3 regime, as the Lagrangian and Eulerian scales are modified.

The basic assumption of the LSM relies on the Markovianity of the velocity process. In fact, in the inertial subrange the Lagrangian acceleration autocorrelation scale is of the order of the Kolmogorov time scale τ_η, which decreases with increasing Reynolds number, as predicted by the Heisenberg-Yaglom formula based on the Kolmogorov theory [4]. This prediction was experimentally confirmed in Ref. [16]. Another experimental support for the LSM, in particular to the WMC, was given in Refs. [17,18], where the Eulerian probability density function (PDF) is shown to fulfil the Chapman-Kolmogorov equation underlying the Markovian assumption. Intermittency effects are not considered because they are found to be negligible in the LSM [19].

Thus, according to Ref. [15], the particle separation $\delta\mathbf{x}$ and velocity difference $\delta\mathbf{v}$ are represented by a stochastic differential equation (SDE) of the Langevin type. Using the above defined scales and a velocity scale υ, to make the model nondimensional, the Langevin equation turns out to be

$$d\delta x_i = \beta \delta v_i dt,$$

$$d\delta v_i = a_i(\delta\mathbf{v}, \delta\mathbf{x}, t)dt + b_{ij}(\delta\mathbf{x}, t)dW_j, \tag{1}$$

where dW_j is a component of a three-dimensional Wiener process and $\beta = \upsilon\tau\lambda^{-1}$ is recognized as the well known Lagrangian-to-Eulerian scale ratio. In Ref. [14] the SDE coefficients a_i and b_{ij} are called the drift and diffusion coefficients, respectively. The associated nondimensional Fokker-Planck equation turns out to be

$$\frac{\partial p}{\partial t} = -\beta \frac{\partial}{\partial \delta x_i}(\delta v_i p) - \frac{\partial}{\partial \delta v_i}(a_i p) + b_{ij}\frac{\partial^2 p}{\partial \delta v_i \partial \delta v_i}, \tag{2}$$

where p is the Lagrangian PDF of the process $(\delta\mathbf{x}, \delta\mathbf{v})$. To avoid physical inconsistencies caused by different Ito and Stratonovich interpretations of the stochastic calculus, the tensor b_{ij} must be independent of $\delta\mathbf{v}$ [20]. For consistency with the Lagrangian structure function of the second order, $b_{ij} = \sqrt{2}\delta_{ij}$. The drift term is retrieved by imposing the Eulerian statistical consistency (WMC) [14], which is obtained by the Novikov relation between Eulerian and Lagrangian PDFs [21] in combination with the Fokker-Planck equation. Under this assumption, it turns out that

$$a_i = \frac{\partial \ln p_E}{\partial \delta v_i} + \Phi_i, \tag{3}$$

where p_E is the Eulerian PDF and

$$\frac{\partial(\Phi_i p_E)}{\partial \delta v_i} = -\frac{\partial p_E}{\partial t} - \beta \frac{\partial(\delta v_i p_E)}{\partial \delta x_i}, \tag{4}$$

with $\Phi_i \to 0$ as $|\delta\mathbf{v}| \to \infty$. It follows that, for a given p_E (based on assumed or empirically specified Eulerian statistics), each solution of Eqs. (3) and (4) depends solely on β, as do all the statistics at all times. It should also be noted that the parameter β is a weight of the effect of the intrinsic inhomogeneity of the relative dispersion process.

Before proceeding with the analysis of model properties, it is worth focusing attention on the definition of the appropriate scales. It can be observed that there is no general agreement [22–25] on the value of the nondimensional parameter β. In fact, if integral measures are considered, the value of β could depend on experimental configuration and on the kind of flow [24]. A more rigorous definition should be related to inertial subrange features. In the following, this ratio will be defined in terms of the Kolmogorov constants C_K and C_0. As the interest here is on inertial subrange properties, specific scales should be used.

Considering a separation r' (hereinafter a prime denotes dimensional variables) in the inertial subrange and a mean rate of energy dissipation ε', the Eulerian second order structure function is [4]

$$\langle \Delta_r u_i' \Delta_r u_j' \rangle = C_K(\varepsilon' r')^{2/3}\delta_{ij} = 2\sigma^2\left(\frac{r'}{\lambda}\right)^{2/3}\delta_{ij},$$

where the second equality is a definition for a Eulerian spatial scale λ, the angular brackets denote an ensemble average, $\Delta_r u_i' = u_i'(\mathbf{x}') - u_i'(\mathbf{x}' + \mathbf{r}')$, $r = |\mathbf{r}'|$, σ is the root mean square of the velocity fluctuations and δ_{ij} is the Kronecker delta. In the inertial subrange, the Lagrangian structure function of the second order is

$$\langle \Delta_t v_i' \Delta_t v_j' \rangle = C_0 \varepsilon' t' \delta_{ij} = 2\sigma^2\left(\frac{t'}{\tau}\right)\delta_{ij},$$

which is similarly a definition for a Lagrangian temporal scale τ.

Selecting the normalizing velocity scale $v = \sqrt{2}\sigma$, the nondimensional parameter β can therefore be expressed as

$$\beta \equiv \frac{\sqrt{2}\sigma\tau}{\lambda} = \frac{C_K^{3/2}}{C_0}, \tag{5}$$

which is equivalent, in homogeneous steady turbulence, to the ratio between integral temporal and spatial scales (see, for instance, Refs. [22–25]). Nevertheless, the definition in Eq. (5) is rather general, because it is not affected by inhomogeneity or unsteadiness at scales greater than the inertial subrange, and does not require the introduction of an advection velocity.

The nondimensional Richardson t^3 law [5] reads

$$\langle (\delta\mathbf{x})^2 \rangle = g^* \beta^2 t^3, \tag{6}$$

where $g^* = g/C_0$. The quantities g^* and β characterize the relative dispersion process. It is interesting to observe that g^* is equal to the ratio between the velocity difference of two particles at fixed time $\langle (\delta v_i)^2 \rangle = g\varepsilon t$, on the one hand (Ref. [4] p. 545), and the Lagrangian structure function of the second order $\langle (\Delta_t v_i)^2 \rangle$, on the otherhand. This ratio turns out to be a constant, insofar as the process of pair dispersion can be regarded as the motion of one particle relative to a noninertial system, whose origin moves with the other particle. However, for small times, $t \ll \tau$, this motion is not very different from the motion relative to an inertial system, whose origin moves at the initial velocity of the same particle (Ref. [4] p. 546). The above observation suggests the adoption of C_0 as a scale for g, so that g^* can be considered as a normalized Richardson constant.

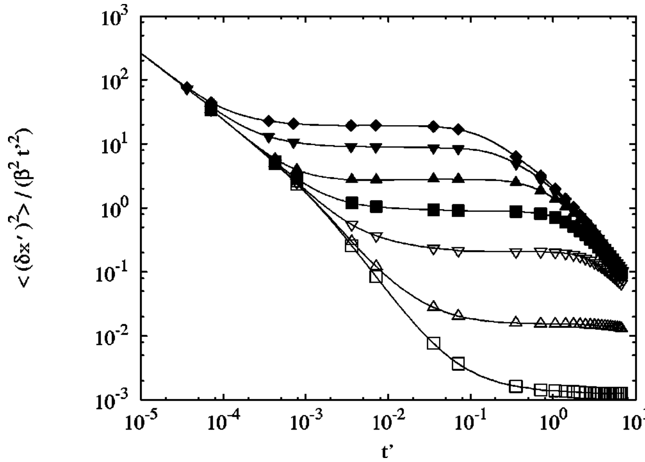

FIG. 1. The nondimensional separation normalized to $\beta^2 t^3$ as a function of t in the cases $\beta=0.1$ (□), 0.2 (△), 0.5 (▽), 1 (■), 2 (▲), 5 (▼), and 10 (◆), at fixed initial separation $\delta x_i'(t_0') = 10^{-5}$.

Although C_K and C_0 are universal constants in the fully developed turbulence frame, their value is still not well determined due to experimental difficulties and errors. Values found in the literature are $1.5 \leqslant C_K \leqslant 3$ [26] and $2 \leqslant C_0 \leqslant 7$ [27].

Considering C_K and C_0 as parameters in a model, variations in the range of their experimental indetermination will affect the model properties and, particularly, the Richardson constant. This reflects the property expressed by Eq. (6). It is worth noting that the dependence of the model only on β clearly highlights the fact that g can be determined when C_K and C_0 are known. In other words, once a model is chosen in the WMC class, it can be supposed that a relationship exists which determines g through C_K and C_0.

To investigate the properties expressed in Eq. (4), numerical integrations of Eq. (1) were performed. Because Eqs. (3) and (4) in general do not determine a_i uniquely (see Ref. [28] for a review), a_i is chosen in the following as in Ref. [15]. This choice corresponds to a Gaussian zero mean random velocity field with longitudinal velocity correlation given by [29]

$$f = 1 - \left(\frac{\delta x^2}{\delta x^2 + 1}\right)^{1/3}.$$

Equations (1) were solved numerically, and the statistics were computed using 10^4 pairs, with the initial relative velocity distributed according to $\langle (\delta v_i'(t_0'))^2 \rangle$

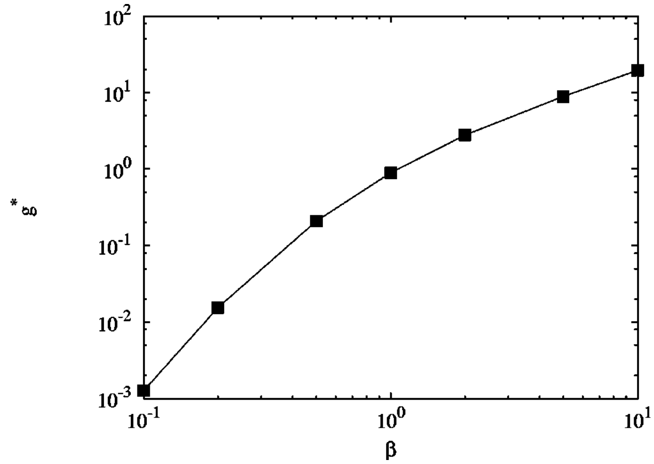

FIG. 2. The normalized Richardson constant vs the parameter β.

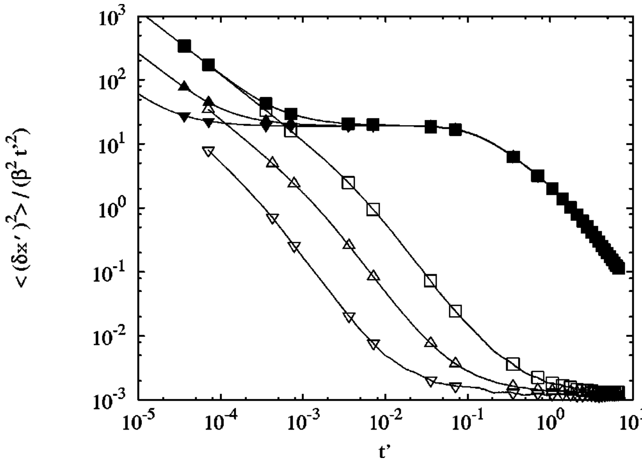

FIG. 3. The nondimensional separation normalized to $\beta^2 t^3$ as a function of t in the cases $\delta x_i(t_0)=10^{-4}$ (□,■), 10^{-5} (△,▲), and 10^{-6} (▽,▼), for β=0.1 and 10, respectively.

$=C_k(\varepsilon'|\delta\mathbf{x}'(t_0')|)^{2/3}$. The analysis was performed for $0.1\leqslant\beta\leqslant 10$, according to the ranges of C_K and C_0 found in the literature.

In Fig. 1 the quantity $\langle(\delta\mathbf{x})^2\rangle/(\beta^2 t^3)$ is plotted as a function of t for different values of β at fixed initial separation $\delta x_i(t_0)=10^{-5}$. The plateau indicates the existence and the extension of the t^3 regime and provides a measure of g^*. The value of g^* increases when β increases, as shown in Fig. 2, since an increasing β corresponds to a weaker Eulerian spatial correlation. Moreover, Fig. 1 shows that the temporal extension of the inertial regime t^3 decreases as β increases. The temporal extension can be estimated as the interval $t_d'-t_s'$, where t_s' is defined as the intersection point between the ballistic regime, $\langle(\delta x_i')^2\rangle=C_k(\varepsilon'|\delta\mathbf{x}'(t_0')|)^{2/3}t'^2$, and the inertial regime, $\langle(\delta x_i')^2\rangle=g\varepsilon' t'^3/3$. The time t_d' is defined as the intersection point between the inertial regime and the diffusive regime, $\langle(\delta x_i')^2\rangle=4\sigma^2 T_L t'$, where T_L is the integral Lagrangian temporal scale. In nondimensional terms, the temporal extension turns out to be

$$t_d-t_s=\sqrt{\frac{6}{g^*}\left(\frac{T_L}{\tau}\right)^{1/2}}-\frac{3}{g^*}(|\delta\mathbf{x}(t_0)|)^{3/2}, \qquad (7)$$

which clearly highlights the dependence of t_d-t_s on $|\delta\mathbf{x}(t_0)|$. Assuming that T_L and τ are of the same order (see, e.g., Ref. [30]) and $|\delta\mathbf{x}(t_0)|\ll 1$, it turns out that $t_d-t_s\simeq\sqrt{6/g^*}$. Figure 3 confirms this result, while also showing that g^* does not depend on the initial separation, in keeping with the physical properties of the t^3 regime [4].

In this Brief Report it is shown that any given Lagrangian stochastic model based on the well mixed condition depends only on the parameter β, the Lagrangian-to-Eulerian scale ratio. This parameter has been formulated here in terms of universal constants characteristic of the inertial subrange, i.e., C_K and C_0. In spite of the fact that these constants are expected to be universal, in the range of error associated with their experimental determination, their variation strongly influences the dispersion properties of the model. It can be observed that the numerical values of g^* and t_d-t_s refer to the specific turbulence model adopted. However, the nondimensional analysis suggests that the same behavior can be expected from the whole class of models based on the WMC assumption. Thus, the present result can be considered to be of general validity within the context of the WMC approach to Lagrangian turbulence modeling.

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