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Probability distributions as solutions to fractional diffusion equations

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Abstract

The fundamental solutions (Green functions) for the Cauchy problem of the space-time fractional diffusion equation are investigated with respect to their scaling and similarity properties, starting from their composite Fourier-Laplace representation. By using the Mellin transform, a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane is presented, that allows us to obtain their computational form in the space-time domain and to analyse their probability interpretation.

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1 Introduction

We consider the Cauchy problem for the *space-time fractional* partial differential equation, which is obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $\alpha \in (0,2]$ and skewness θ ($|\theta| \le \min\{\alpha, 2-\alpha\}$), and the first-order time derivative with a Caputo derivative of order $\beta \in (0,2]$. The fundamental solutions (Green functions) for the Cauchy problem are investigated with respect to their scaling and similarity properties, starting from their combined Fourier-Laplace representation.

In the cases $\{0 < \alpha \le 2\,, \beta = 1\}$ and $\{\alpha = 2\,, 0 < \beta \le 1\}$ the fundamental solutions are known to be interpreted as a *spatial probability density functions evolving in time*, so we talk of *space-fractional diffusion* and *time-fractional diffusion*, respectively. Then, by using the Mellin transform, we provide a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane, which allows us *to extend the probability interpretation to the ranges* $\{0 < \alpha \le 2\,, 0 < \beta \le 1\}$ *and* $\{1 < \beta \le \alpha \le 2\}$. Furthermore, from this representation it is possible to derive explicit formulae (convergent series and asymptotic expansions), which enable us to plot the spatial probability densities for different values of the relevant parameters α, θ, β.

2 The space-time fractional diffusion equation

By replacing in the standard diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{2.1}$$

where $u = u(x,t)$ is the (real) field variable, the second-order space derivative and the first-order time derivative by suitable *integro-differential* operators, which can be interpreted as a space and time derivative of fractional order, we obtain a sort of "generalized diffusion" equation. Such equation may be referred to as the *space-time fractional diffusion* equation when its fundamental solution (see below) can be interpreted as a probability density. We write

$$_tD_*^\beta\, u(x,t) = {_xD_\theta^\alpha}\, u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{2.2}$$

where the α, θ, β are real parameters restricted as follows

$$0 < \alpha \leq 2\,, \quad |\theta| \leq \min\{\alpha, 2-\alpha\}\,, \quad 0 < \beta \leq 2\,. \tag{2.3}$$

In (2.2) $_xD_\theta^\alpha$ is the *Riesz-Feller fractional derivative* (in space) of order α and skewness θ, and $_tD_*^\beta$ is the *Caputo fractional derivative* (in time) of order β. The definitions of these fractional derivatives are more easily understood if given in terms of Fourier transform and Laplace transform, respectively.

For the *Riesz-Feller fractional derivative* we have

$$\mathcal{F}\{{_xD_\theta^\alpha}\, f(x); \kappa\} = -\psi_\alpha^\theta(\kappa)\, \widehat{f}(\kappa)\,, \quad \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha\, e^{i(\text{sign}\,\kappa)\theta\pi/2}\,, \tag{2.4}$$

where $\kappa \in \mathbb{R}$ and $\widehat{f}(\kappa) = \mathcal{F}\{f(x);\kappa\} = \int_{-\infty}^{+\infty} e^{+i\kappa x} f(x)\, dx$. In other words the symbol of the pseudo-differential operator[1] $_xD_\theta^\alpha$ is required to be the logarithm of the characteristic function of the generic *stable* (in the Lévy sense) probability density, according to the Feller parameterization [6], [7].

For $\alpha = 2$ (hence $\theta = 0$) we have $\widehat{{_xD_0^2}}(\kappa) = -\kappa^2 = (-i\kappa)^2$, so we recover the standard second derivative.

For $0 < \alpha < 2$ and $\theta = 0$ we have $\widehat{{_xD_0^\alpha}}(\kappa) = -|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$ so

$$_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}. \tag{2.5}$$

In this case we call the LHS of (2.5) simply the *Riesz fractional derivative* of order α. For the explicit expressions in integral form of the general *Riesz-Feller fractional derivative* we refer the interested reader *e.g.* to [13], [15], [25], [34].

Let us now consider the *Caputo fractional derivative.* Following the original idea by Caputo [2], see also [3], [12], [32], a proper time fractional derivative of order $\beta \in (m-1, m]$ with $m \in \mathbb{N}$, useful for physical applications, may be defined in terms of the following rule for the Laplace transform:

$$\mathcal{L}\{{_tD_*^\beta}\, f(t); s\} = s^\beta\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k} f^{(k)}(0^+)\,, \quad m-1 < \beta \leq m\,, \tag{2.6}$$

where $s \in \mathbb{C}$ and $\widetilde{f}(s) = \mathcal{L}\{f(t);s\} = \int_0^\infty e^{-st} f(t)\, dt$. Then the *Caputo fractional derivative* of $f(t)$ turns out to be

$$_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\beta+1-m}}\,, & m-1 < \beta < m\,, \\[2ex] \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,, \end{cases} \quad t \geq 0\,. \tag{2.7}$$

[1] Let us recall that a generic linear pseudo-differential operator A, acting with respect to the variable $x \in \mathbb{R}$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} e^{i\kappa x} A\,[f(x)]\, dx = \widehat{A}(\kappa)\, \widehat{f}(\kappa)$, where $\widehat{A}(\kappa)$ is referred to as symbol of A, given as $\widehat{A}(\kappa) = \left(A\, e^{-i\kappa x}\right) e^{+i\kappa x}$.

In order to formulate and solve the Cauchy problems for (2.2) we have to select explicit initial conditions concerning $u(x,0^+)$ if $0 < \beta \leq 1$ and $u(x,0^+)$, $u_t(x,0^+)$ if $1 < \beta \leq 2$. If $\phi_1(x)$ and $\phi_2(x)$ denote two given real functions of $x \in \mathbb{R}$, the Cauchy problems consist in finding the solution of (2.2) subjected to the additional conditions:

$$u(x,0^+) = \phi_1(x)\,, \quad x \in \mathbb{R}\,, \quad \text{if} \quad 0 < \beta \leq 1\,; \tag{2.8a}$$

$$\begin{cases} u(x,0^+) = \phi_1(x)\,, \\ u_t(x,0^+) = \phi_2(x)\,, \end{cases} \quad x \in \mathbb{R}\,, \quad \text{if} \quad 1 < \beta \leq 2\,. \tag{2.8b}$$

3 Representations of the Green functions

The Cauchy problems can be conveniently treated by making use of the most common integral transforms, *i.e.* the Fourier transform (in space) and the Laplace transform (in time). The composite Fourier-Laplace transforms of the solutions of the two Cauchy problems turn out to be, by using (2.4) and (2.6) with $m = 1, 2$,

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}\, \widehat{\phi_1}(\kappa)\,, \quad 0 < \beta \leq 1\,, \tag{3.1a}$$

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}\, \widehat{\phi_1}(\kappa) + \frac{s^{\beta-2}}{s^\beta + \psi_\alpha^\theta(\kappa)}\, \widehat{\phi_2}(\kappa)\,, \quad 1 < \beta \leq 2\,. \tag{3.1b}$$

By fundamental solutions (or Green functions) of the above Cauchy problems we mean the (generalized) solutions corresponding to the initial conditions:

$$G_{\alpha,\beta}^{\theta\,(1)}(x,0^+) = \delta(x)\,, \quad 0 < \beta \leq 1\,; \tag{3.2a}$$

$$\begin{cases} G_{\alpha,\beta}^{\theta\,(1)}(x,0^+) = \delta(x)\,, \\ \dfrac{\partial}{\partial t} G_{\alpha,\beta}^{\theta\,(1)}(x,0^+) = 0\,, \end{cases} \quad \begin{cases} G_{\alpha,\beta}^{\theta\,(2)}(x,0^+) = 0\,, \\ \dfrac{\partial}{\partial t} G_{\alpha,\beta}^{\theta\,(2)}(x,0^+) = \delta(x)\,, \end{cases} \quad 1 < \beta \leq 2\,. \tag{3.2b}$$

We have denoted by $\delta(x)$ the delta-Dirac generalized function, whose (generalized) Fourier transform is known to be 1, and we have distinguished by the apices (1) and (2) the two types of Green functions. From Eqs (3.1a)-(3.1b) the composite Fourier-Laplace transforms of these Green functions turn out to be

$$\widehat{\widetilde{G_{\alpha,\beta}^{\theta\,(j)}}}(\kappa, s) = \frac{s^{\beta-j}}{s^\beta + \psi_\alpha^\theta(\kappa)}\,, \quad 0 < \beta \leq 2\,, \quad j = 1, 2\,. \tag{3.3}$$

Furthermore, by recalling the Fourier convolution property, we note that the Green functions allow us to represent the solutions of the above two Cauchy problems through the relevant integral formulas:

$$u(x,t) = \int_{-\infty}^{+\infty} G_{\alpha,\beta}^{\theta\,(1)}(\xi,t)\, \phi_1(x-\xi)\, d\xi\,, \quad 0 < \beta \leq 1\,; \tag{3.4a}$$

$$u(x,t) = \int_{-\infty}^{+\infty} \left[G_{\alpha,\beta}^{\theta\,(1)}(\xi,t)\phi_1(x-\xi) + G_{\alpha,\beta}^{\theta\,(2)}(\xi,t)\phi_2(x-\xi)\right] d\xi, \; 1 < \beta \leq 2. \tag{3.4b}$$

We recognize from (3.3) that the function $G_{\alpha,\beta}^{\theta\,(2)}(x,t)$ along with its Fourier-Laplace transform is well defined also for $0 < \beta \leq 1$ even if it loses its meaning of being a fundamental solution of (3.2), resulting

$$G_{\alpha,\beta}^{\theta\,(2)}(x,t) = \int_0^t G_{\alpha,\beta}^{\theta\,(1)}(x,\tau)\, d\tau\,, \quad 0 < \beta \leq 2\,. \tag{3.5}$$

By using the known scaling rules for the Fourier and Laplace transforms, and introducing the *similarity variable* $x/t^{\beta/\alpha}$, we infer from (3.3) (thus without inverting the two transforms) the *scaling properties* of the Green functions,

$$G^{\theta\,(j)}_{\alpha,\beta}(x,t) = t^{-\beta/\alpha+j-1}\, K^{\theta\,(j)}_{\alpha,\beta}\left(x/t^{\beta/\alpha}\right), \quad j=1,2, \tag{3.6}$$

where the one-variable functions $K^{\theta\,(j)}_{\alpha,\beta}(x)$, obtained by setting $t=1$, are called the *reduced Green functions*. We also note the *symmetry relation*:

$$G^{\theta\,(j)}_{\alpha,\beta}(-x,t) = G^{-\theta\,(j)}_{\alpha,\beta}(x,t), \quad j=1,2, \tag{3.7}$$

so for the determination of the Green functions we can restrict our attention to $x>0$. Extending the method illustrated in [9], [25], where only the Green function of type (1) was determined, we first invert the Laplace transforms in (3.3) getting

$$\widehat{G^{\theta\,(j)}_{\alpha,\beta}}(\kappa,t) = t^{j-1}\, E_{\beta,j}[-\psi^\theta_\alpha(\kappa)t^\beta],\ \widehat{K^{\theta\,(j)}_{\alpha,\beta}}(\kappa) = E_{\beta,j}[-\psi^\theta_\alpha(\kappa)],\ j=1,2, \tag{3.8}$$

where $E_{\beta,j}$ denotes the two-parameter Mittag-Leffler function[2]. We note the normalization property satisfied by both reduced Green functions: $\int_{-\infty}^{+\infty} K^{\theta\,(j)}_{\alpha,\beta}(x)\,dx = E_{\beta,j}(0) = 1/\Gamma(j) = 1$ for $j=1,2$. However, the normalization property holds true for all times only for the first complete Green function as we can note from the first equality in (3.8). Following [25] we invert the Fourier transforms of $K^{\theta\,(j)}_{\alpha,\beta}(x)$ by using the convolution theorem of the Mellin transforms arriving at the Mellin-Barnes integral representation

$$K^{\theta\,(j)}_{\alpha,\beta}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})\,\Gamma(1-s)}{\Gamma(j-\frac{\beta}{\alpha}s)\,\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\, x^{\,s}\, ds, \tag{3.9}$$

where $0<\gamma<\min\{\alpha,1\}$, and $\rho=(\alpha-\theta)/(2\,\alpha)$.

For later use we recall the main formulas concerning the Mellin transform. For more details, see *e.g.* [27]. If

$$\mathcal{M}\{f(r);s\} = f^*(s) = \int_0^{+\infty} f(r)\,r^{s-1}\,dr, \quad \gamma_1 < \Re(s) < \gamma_2 \tag{3.10}$$

denotes the Mellin transform of $f(r)$ with $r\in\mathbb{R}^+$, the inversion is provided by

$$\mathcal{M}^{-1}\{f^*(s);r\} = f(r) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)\,r^{-s}\,ds, \tag{3.11}$$

where $r>0$, $\gamma=\Re(s)$, $\gamma_1<\gamma<\gamma_2$. The Mellin convolution formula reads

$$h(r) = \int_0^\infty \frac{1}{\rho}\, f(\rho)\, g(r/\rho)\,d\rho \overset{\mathcal{M}}{\leftrightarrow} h^*(s) = f^*(s)\,g^*(s). \tag{3.12}$$

We note that the Mellin-Barnes integral representation (3.9)[3] allows us to construct computationally the fundamental solutions of Eq. (3.2) for any triplet $\{\alpha,\beta,\theta\}$ by matching their convergent

[2]The Mittag-Leffler function $E_{\beta,\mu}(z)$ with $\beta,\mu>0$ is an entire transcendental function of order $\rho=1/\beta$, defined in the complex plane by the power series

$$E_{\beta,\mu}(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta\, n+\mu)}, \quad \beta,\mu>0, \quad z\in\mathbb{C}.$$

For information on the Mittag-Leffler-type functions the reader may consult *e.g.* [5], [12], [32].

[3]The names refer to the two authors, who in the beginning of the past century developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, as revisited in [26], these integrals were first introduced in 1888 by S. Pincherle (Professor of Mathematics at the University of Bologna from 1880 to 1928).
As a matter of fact this type of integrals turns out to be useful in inverting the Mellin transforms.
Readers acquainted with Fox H functions can recognize in (3.9) the representation of a certain function of this class, see *e.g.* [28], [37]. Unfortunately, as far as we know, computing routines for this general class of special functions are not yet available.

and asymptotic expansions, as shown in [25] for the first Green function. The interested reader may take vision of several plots of the *reduced* Green functions in [25] in a number of cases where these functions, being non-negative and normalized, can be interpreted as probability densities. In order to give the reader a better impression about the behaviours of the tails, the logarithmic scale was adopted.

We also note that the space-time fractional diffusion equation has been analysed (but without numerical computations) in several papers, see *e.g.* Anh and Leonenko [1] and references therein.

4 Probability interpretation of the Green functions

For the following cases that allow simplifications in the integrand of Eq. (3.9), we obtain relevant expressions of the corresponding Green functions that can be interpreted as probability densities.
(a) For $j=1$ and $\{0<\alpha<2,\ \beta=1\}$ (*strictly space fractional diffusion*) we have $K^{\theta\,(1)}_{\alpha,1}(x) = L^\theta_\alpha(x)$, *i.e.* the class of the strictly stable (non-Gaussian) probability densities [7][4]exhibiting fat tails (with the algebraic decay $\propto |x|^{-(\alpha+1)}$) and infinite variance. Their Mellin-Barnes integral representation reads

$$K^{\theta\,(1)}_{\alpha,1}(x) = L^\theta_\alpha(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s/\alpha)\,\Gamma(1-s)}{\Gamma(\rho\, s)\,\Gamma(1-\rho s)}\,x^{\,s}\,ds, \tag{4.1}$$

where $0<\gamma<\min\{\alpha,1\}$.
(b) For $j=1,2$ and $\{\alpha=2,\ 0<\beta<2\}$ (*time fractional diffusion* including *standard diffusion*), we have $K^{0\,(j)}_{2,\beta}(x) = M^{(j)}_{\beta/2}(x)/2$, *i.e.* the class of the Wright type[5] probability densities exhibiting stretched exponential tails. Their Mellin-Barnes integral representation reads

$$K^{0\,(j)}_{2,\beta}(x) = \frac{1}{2}M^{(j)}_{\beta/2}(x) = \frac{1}{2x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(1-s)}{\Gamma(j-\beta s/2)}\,x^{\,s}\,ds, \tag{4.2}$$

where $0<\gamma<1$.
(c) For $j=1$ and $\{0<\alpha=\beta<2\}$ (*neutral fractional diffusion*), we have $K^{\theta\,(1)}_{\alpha,\alpha}(x) = N^\theta_\alpha(x)$, *i.e.* the class of the Cauchy type probability densities [25]. Indeed, in this special case, the Mellin-Barnes integral representation provides an explicit expression which generalizes the Cauchy density,

$$\begin{aligned} K^{\theta\,(1)}_{\alpha,\alpha}(x) = N^\theta_\alpha(x) &= \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})}{\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\,x^{\,s}\,ds \\ &= \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\sin(\pi\,\rho\, s)}{\sin(\pi\, s/\alpha)}\,x^{\,s}\,ds = \frac{1}{\pi}\,\frac{x^{\alpha-1}\sin[\frac{\pi}{2}(\alpha-\theta)]}{1+2x^\alpha\cos[\frac{\pi}{2}(\alpha-\theta)]+x^{2\alpha}}. \end{aligned} \tag{4.3}$$

[4]For recent treatises on Lévy stable distributions see *e.g.* [20], [35], [36], [38].

[5]The function $M^{(j)}_\nu(z)$ is defined for any order $\nu\in(0,1)$ and $\forall z\in\mathbb{C}$ by

$$M^{(j)}_\nu(z) := \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\,\Gamma[-\nu n+(j-\nu)]}, \quad 0<\nu<1, \quad z\in\mathbb{C}.$$

It turns out that $M^{(j)}_\nu(z)$ is an entire function of order $\rho=1/(1-\nu)$. For $\nu=1/2$ we obtain

$$M^{(1)}_{1/2}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-z^2/4\right), \quad M^{(2)}_{1/2}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-z^2/4\right) - \frac{z}{2}\mathrm{erfc}\left(\frac{z}{2}\right).$$

The M functions are special cases of the Wright function defined by the series representation, valid in the whole complex plane,

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)}, \quad \lambda>-1, \quad \mu\in\mathbb{C}, \quad z\in\mathbb{C}.$$

Indeed, we recognize $M^{(j)}_\nu(z) = \Phi_{-\nu,j-\nu}(-z)$, $0<\nu<1$. Originally, Wright introduced and investigated this function with the restriction $\lambda\ge 0$ in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions. Only later, in 1940, he considered the case $-1<\lambda<0$. For detailed information on the Wright-type functions the interested reader may consult, *e.g.* [5] (where, presumably for a misprint, λ is restricted to be non-negative), [10], [11], [21].

where $0 < \gamma < \alpha$.

Based on the arguments outlined in [25], we extend the meaning of probability density to the cases $\{0 < \alpha < 2\,, 0 < \beta < 1\}$ and $\{1 < \beta \le \alpha < 2\}$ by proving the following composition rules of the Mellin convolution type:

$$K_{\alpha,\beta}^{\theta\,(j)}(x) = \begin{cases} \alpha \displaystyle\int_0^\infty \left[\xi^{\alpha-1}\, M_\beta^{(j)}\,(\xi^\alpha)\right]\, L_\alpha^\theta\,(x/\xi)\, \frac{d\xi}{\xi}\,, & 0 < \beta < 1\,, \\ \displaystyle\int_0^\infty M_{\beta/\alpha}^{(j)}(\xi)\, N_\alpha^\theta(x/\xi)\, \frac{d\xi}{\xi}\,, & 0 < \beta/\alpha < 1\,. \end{cases} \tag{4.4}$$

The absolute moments of $K_{\alpha,\beta}^{\theta\,(j)}(x)$ can be obtained by considering the Mellin transform of $x\,K_{\alpha,\beta}^{\theta\,(j)}(x)$ which reads, by using (3.9) and (3.10)-(3.11),

$$\int_0^{+\infty} K_{\alpha,\beta}^{\theta\,(j)}(x)\, x^{\,s}\, dx = \rho\, \frac{\Gamma(1 - s/\alpha)\,\Gamma(1 + s/\alpha)\,\Gamma(1+s)}{\Gamma(1-\rho\, s)\,\Gamma(1+\rho\, s)\,\Gamma(j + \beta\, s/\alpha)}\,, \tag{4.5}$$

where $-\min\{\alpha, 1\} < \Re(s) < \alpha$. In particular we find $\int_0^{+\infty} K_{\alpha,\beta}^{\theta\,(j)}(x)\, dx = \rho$ (with $\rho = 1/2$ if $\theta = 0$). We note that Eq. (4.5) is strictly valid as soon as cancellations in the "gamma fraction" at the RHS are not possible. Then this equation allows us to evaluate (in $\mathbb{R}_0^+$) the (absolute) moments of order δ for the Green function if $-\min\{\alpha, 1\} < \delta < \alpha$. In other words, it states that $K_{\alpha,\beta}^{\theta\,(j)}(x) = \mathcal{O}\left(x^{-(\alpha+1)}\right)$ as $x \to +\infty$. However, cancellations occur in the following cases where the restriction $\delta < \alpha$ is expected to disappear:

a) $\{\alpha = 2, \theta = 0, 0 < \beta < 2\}$ (*time fractional diffusion* including *standard diffusion*), for which $\rho = 1/2$;

b) $\{1 < \alpha < 2\,, \theta = \alpha - 2,\, 0 < \beta < \alpha\}$ (*extremal diffusion*), for which $\rho = 1/\alpha$. We note that this may happen only for one tail of the extremal density.

We recognize that case a) is included in case b) in the limit $\alpha = 2$. In the above cases Eq. (4.5) reduces to

$$\int_0^{+\infty} K_{\alpha,\beta}^{2-\alpha\,(j)}(x)\, x^{\,s}\, dx = \frac{1}{2}\, \frac{\Gamma(1+s)}{\Gamma[j + \beta\, s/\alpha]}\,, \quad \Re(s) > -1\,, \tag{4.6}$$

and consequently any absolute moment of order $\delta > -1$ is finite. We can show that the corresponding Green functions result of the Wright type[6] and exhibit a stretched exponential decay according to the asymptotic representation

$$K_{\alpha,\beta}^{2-\alpha\,(j)}(x) \sim \alpha^{-1}\,[2\pi(1-\beta/\alpha)]^{-1/2}(x\beta/\alpha)^{(1/2-j+\beta/\alpha)/(1-\beta/\alpha)} \cdot \exp\left[-(\alpha/\beta - 1)\,(x\beta/\alpha)^{1/(1-\beta/\alpha)}\right]\,, \quad x \to +\infty\,. \tag{4.7}$$

Then, due to the previous discussion, in the cases $\{0 < \alpha < 2\,, 0 < \beta < 1\}$ and $\{1 < \beta \le \alpha < 2\}$ (*i.e. strictly space-time-fractional diffusion*) we obtain a class of probability densities (symmetric or non-symmetric according to $\theta = 0$ or $\theta \neq 0$) which exhibit fat tails (only one fat tail in the extremal cases) with an algebraic decay $\propto |x|^{-(\alpha+1)}$. Thus, they belong to the domain of attraction of the Lévy stable densities of index α and can be referred to as *fractional stable densities*.

When the time variable is considered, in all above cases the first Green function evolves in time as a probability density because it keeps the normalization. The integral over all of $\mathbb{R}$ of the first Green function is independent of time whereas that of the second Green function increases linearly with time.

[6] We have

$$K_{\alpha,\beta}^{2-\alpha\,(j)}(x) = \frac{1}{\alpha}\,\Phi_{-\beta/\alpha, j-\beta/\alpha}(-x) = \frac{1}{\alpha}\sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma[-n\,\beta/\alpha + (j-\beta/\alpha)]}\,.$$

5 Conclusions and outlook

In this paper we have summarized our approach to obtain the fundamental solutions of fractional diffusion equations and have shown how they can be interpreted as probability densities evolving in time.

In recent years evolution equations containing fractional derivatives have gained revived interest in that they are expected to provide suitable mathematical models for describing phenomena of anomalous diffusion and transport dynamics in complex systems, see *e.g.* [4], [19], [23], [24], [29], [31], [33], [38]. and references therein. We point out the fact that all these fractional evolution equations can be considered as master equations for random walk models that turn out to be beyond the classical Brownian motion, see *e.g.* Klafter *et al.* [22]. For a recent review we refer the reader to Metzler and Klafter [30]. Gorenflo and collaborators, see *e.g.* [8], [13], [14], [15], [17], [18], have recently proposed a variety of models of random walk, discrete or continuous in space and time, suitable for simulating fractional diffusion processes.

In [16] Gorenflo and Mainardi have shown how to obtain the space-time fractional diffusion equation (2.1), in the case $0 < \beta \le 1$, $\theta = 0$, by a properly scaled transition to the limit from a general master equation.

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