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MELLIN CONVOLUTION FOR SUBORDINATED STABLE PROCESSES*

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1. Introduction

In recent years, a number of papers have appeared on subordinated stochastic processes in view of their relevance in physical and financial applications (see, e.g., [8, 12]). In brief, according to [2], a subordinated process $X(t) = Y(T(t))$ is obtained by randomizing the time clock of a stochastic process $Y(t)$ using a new clock $T(t)$, where $T(t)$ is a random process with nonnegative independent increments. The resulting process $Y(T(t))$ is said to be subordinated to $Y(t)$, called the parent process, and is directed by $T(t)$, called the directing process. The directing process is often referred to as the randomized time or operational time. In particular, assuming $Y(\tau)$ to be a Markov process with a spatial probability density function (p.d.f.) of x, evolving in time τ, $q_\tau(x) \equiv q(x;\tau)$, and $T(\tau)$ to be a process with nonnegative independent increments with p.d.f. on τ depending on a parameter t, $u_t(\tau) \equiv u(\tau;t)$, then the subordinated process $X(t) = Y(T(t))$ is governed by the spatial p.d.f. on x evolving with t, $p_t(x) \equiv p(x;t)$, given by the integral representation

$$p_t(x) = \int_0^\infty q_\tau(x) u_t(\tau)\, d\tau. \tag{1}$$

If the parent process $Y(\tau)$ is self-similar, namely, its p.d.f. $q_\tau(x)$ is such that

$$q_\tau(x) \equiv q(x;\tau) = \tau^{-1/\alpha} q\left(\frac{x}{\tau^{1/\alpha}}\right), \tag{2}$$

then (1) reads

$$p_t(x) = \int_0^\infty q_\tau\left(\frac{x}{\tau^{1/\alpha}}\right) u_t(\tau)\, \frac{d\tau}{\tau^{1/\alpha}}. \tag{3}$$

In this paper, we shall show how to interpret (3) by means of the convolution integral of the Mellin transform, and how to use the tools of the Mellin–Barnes integrals to treat the subordination for the class of self-similar stochastic processes, which are governed by Lévy strictly stable probability distributions. The paper is divided as follows.

In Sec. 2, we provide the reader with the essential notions and notations concerning the Mellin transforms. In particular, we recall the fundamental property of the Mellin convolution to represent the p.d.f. of the product of two independent random variables and consequently to interpret the subordination formula involving a self-similar parent process.

In Sec. 3, we first recall the Mellin–Barnes integral representation of the whole class of Lévy (symmetric and not symmetric) strictly stable densities, as derived in previous papers [3, 5]. Then, through the Mellin convolution, we derive for these densities a subordination formula involving, as directing p.d.f.'s, those of unilateral stable densities. In so doing, we clarify and generalize known results of Feller, proven via Fourier and Laplace transforms.

Finally, in Sec. 4, we draw the main conclusions and outline the direction for future work.

2. The Mellin Transform

For the reader's convenience, we present here, using our notations, an introduction to the Mellin transform, which turns out to play a basic role in our treatment of the subordination.

Since, in what follows, we shall meet mostly real (or complex-valued) functions of a real variable that are defined and continuous in a given open interval $\mathcal{I} = (a, b)$, $-\infty \le a < b \le +\infty$, except, possibly, at isolated points where these functions can be infinite, we restrict our presentation of the Mellin transform to the class of functions for which the Riemann improper integral on $\mathcal{I}$ absolutely converges. In so doing, we follow Marichev [7], and we denote this class by $L^c(\mathcal{I})$ or $L^c(a, b)$.

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Let

$$\mathcal{M}\{f(x);s\} = f^*(s) = \int\limits_0^\infty f(x)x^{s-1}\,dx, \quad \gamma_1 < \Re(s) < \gamma_2, \tag{4}$$

the Mellin transform of a (sufficiently well-behaved) function $f(x) \in L^c(0,+\infty)$, and

$$\mathcal{M}^{-1}\{f^*(s);x\} = f(x) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{-s}\,ds, \quad \gamma_1 < \gamma < \gamma_2, \tag{5}$$

be the inverse Mellin transform.[1]

We refer to specialized treatises and/or handbooks (see, e.g., [1, 7]) for more details and tables on the Mellin transform. Here we recall the main rules that are relevant in the following.

Denoting by $\stackrel{\mathcal{M}}{\longleftrightarrow}$ the juxtaposition of a function $f(x)$ on $x > 0$ with its Mellin transform $f^*(s)$, we have

$$\begin{aligned} x^a f(x) &\stackrel{\mathcal{M}}{\longleftrightarrow} f^*(s+a), \quad a \in \mathbf{C}, \\ f(x^b) &\stackrel{\mathcal{M}}{\longleftrightarrow} \frac{1}{|b|} f^*(s/b), \quad b \in \mathbf{C}, \quad b \neq 0, \\ f(cx) &\stackrel{\mathcal{M}}{\longleftrightarrow} c^{-s} f^*(s), \quad c \in \mathbf{R}, \quad c > 0, \end{aligned}$$

from which

$$x^a f(cx^b) \stackrel{\mathcal{M}}{\longleftrightarrow} \frac{1}{|b|} c^{-(s+a)/b} f^*\left(\frac{s+a}{b}\right). \tag{6}$$

Furthermore, we have

$$h(x) = \int\limits_0^\infty f\left(\frac{x}{\xi}\right) g(\xi)\,\frac{d\xi}{\xi} \stackrel{\mathcal{M}}{\longleftrightarrow} f^*(s)g^*(s) = h^*(s), \tag{7}$$

which is known as the Mellin convolution formula.[2]

The Mellin convolution formula plays a noteworthy role in the algebra of random variables, since the probability density of the product of two independent random variables is given by the Mellin convolution of the two corresponding densities (see, e.g., [11, 14]). For use of this fundamental property in the following, we find it convenient to sketch hereafter its proof, by using a proper notation adapted for our purposes.

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[1] For the existence of the Mellin transform and the validity of the inversion formula, we need to recall the following theorems, adapted from Marichev's treatise (see [7, Theorems 11 and 12, p. 39]).
(a) Let $f(x) \in L^c(\varepsilon, E)$, $0 < \varepsilon < E < \infty$, be continuous in the intervals $(0,\varepsilon]$ and $[E,\infty)$, and let $|f(x)| \le Mx^{-\gamma_1}$ for $0 < x < \varepsilon$ and $|f(x)| \le Mx^{-\gamma_2}$ for $x > E$, where M is a constant. Then, for the existence of a strip in the s-plane in which $f(x)x^{s-1}$ belongs to $L^c(0,\infty)$, it is sufficient that $\gamma_1 < \gamma_2$. When this condition holds, the Mellin transform $f^*(s)$ exists and is analytic in the vertical strip $\gamma_1 < \gamma = \Re(s) < \gamma_2$.
(b) If $f(x)$ is piecewise differentiable and $f(x)x^{\gamma-1} \in L^c(0,\infty)$, then (5) holds true at all points where $f(x)$ is continuous and the (complex) integral in it must be understood in the sense of the Cauchy principal value.

[2] Here we recall the proof of (7). Writing

$$h^*(s) = \int\limits_0^\infty \left[\int\limits_0^\infty f\left(\frac{x}{\xi}\right) g(\xi)\,\frac{d\xi}{\xi}\right] x^{s-1}\,dx = \int\limits_0^\infty g(\xi) \left[\int\limits_0^\infty f\left(\frac{x}{\xi}\right) x^{s-1}\,dx\right] \frac{d\xi}{\xi}$$

and carrying out the change of variables in the inner integral $x \to \eta := x/\xi$ so that $x = \xi\eta$ and $dx = \xi\,d\eta$, we finally have

$$h^*(s) = \int\limits_0^\infty g(\xi)\left[\int\limits_0^\infty f(\eta)(\xi\eta)^{s-1}\xi\,d\eta\right]\frac{d\xi}{\xi} = \int\limits_0^\infty g(\xi)\xi^{s-1}\,d\xi \int\limits_0^\infty f(\eta)\eta^{s-1}\,d\eta = g^*(s)f^*(s) = f^*(s)g^*(s).$$

Let T_1 and T_2 be two nonnegative real independent random variables with p.d.f.'s $p_1(\tau_1)$ and $p_2(\tau_2)$, respectively, $\tau_1, \tau_2 \in \mathbf{R}^+$. We have

$$p(\tau_1, \tau_2) = p_1(\tau_1) p_2(\tau_2), \quad \tau_i \in \mathbf{R}^+, \quad i = 1, 2. \tag{8}$$

Denoting by X the random variable obtained by the product of T_1 and T_2, i.e., $x = \tau_1 \tau_2$, and carrying out the transformation

$$\begin{cases} \tau_1 = x/\tau, \\ \tau_2 = \tau, \end{cases} \tag{9}$$

we get

$$p(x, \tau)\, dx\, d\tau = p_1(x/\tau) p_2(\tau) |J|\, dx\, d\tau, \tag{10}$$

where J is Jacobian of the transformation (9). Noting that $|J| = 1/\tau$ and integrating (10) in $d\tau$, we finally get the p.d.f. of X,

$$p(x) = \int_{\mathbf{R}^+} p_1(x/\tau) p_2(\tau) \frac{d\tau}{\tau}. \tag{11}$$

Comparing with (7), we recognize on the RHS of (11) the Mellin convolution between the p.d.f.'s $p_1(x)$ and $p_2(x)$.

We now adapt (11) to our subordination formula (3) by identifying p_1 and p_2 with q and u, respectively, and carrying out the transformation

$$\begin{cases} \tau_1 = x/\tau^{1/\alpha}, \\ \tau_2 = \tau. \end{cases} \tag{12}$$

Then (11) reads

$$p(x) = \int_0^\infty q(x/\tau^{1/\alpha}) u(\tau) \frac{d\tau}{\tau^{1/\alpha}}. \tag{13}$$

Thus, we recognize the identity of (13) with (3) provided that we label the p.d.f.'s q and u with the subscripts τ and t, respectively. Furthermore, by virtue of the property of the Mellin convolution with respect to the product of two independent random variables (see (11)), we can now interpret the subordination formula (3) as follows. The p.d.f. of the subordinated process X, $p_t(x)$, turns out to be the p.d.f. of the product of the independent random variables X_q and ${X_u}^{1/\alpha}$ distributed according to $q_\tau(x_q)$ and $u_t(x_u)$, respectively.

3. The Subordination for Stable Processes

In Feller's work (see [2, p. 176]), we read "Let X and Y be independent strictly stable variables with characteristic exponent α and β, respectively. Assume Y to be a positive variable (whence $\beta < 1$). The product $XY^{1/\alpha}$ has a stable distribution with exponent $\alpha\beta$."

In other words, in view of the final consideration of the last section, this statement means that any strictly stable process (of exponent $\gamma = \alpha\beta$) is subordinated to a parent strictly stable process (of exponent α) and directed by a unilateral strictly stable process (of exponent $\beta < 1$). Feller's proof is vague, being, as a matter of fact, limited to symmetric subordinated and parent stable distributions. Furthermore, the proof, scattered in several sections, is essentially based on the use of Fourier and Laplace transforms. Here we would like to make more precise the previous statement by Feller by considering the possibility of asymmetry characterized by an index θ as explained in the following. We also find it convenient to use the Mellin machinery by recalling the so-called Mellin–Barnes integral representation of the strictly stable densities. We essentially refer to the notations of the paper [5], where these densities are denoted by

$$L_\alpha^\theta(x), \quad 0 < \alpha \le 2, \quad |\theta| \le \min\{\alpha, 2 - \alpha\}, \quad x \in \mathbf{R}, \tag{14}$$

where α is the stability exponent and θ is a real parameter related to the asymmetry. This notation is justified by our canonical form of the characteristic function for strictly stable densities. If

$$\widehat{f}(\kappa) = \mathcal{F}\{f(x); \kappa\} = \int_{-\infty}^{+\infty} \exp\{+i\kappa x\} f(x)\, dx, \quad \kappa \in \mathbf{R},$$

is the Fourier transform of a function $f(x) \in L^c(\mathbf{R})$, our canonical form reads

$$\widehat{L_\alpha^\theta}(\kappa) = \exp[-|\kappa|^\alpha \exp\{i (\operatorname{sign} \kappa) \theta \pi/2\}]. \tag{15}$$

We note the symmetry relation

$$L_\alpha^\theta(-x) = L_\alpha^{-\theta}(x), \tag{16}$$

which allows us to restrict our attention from now on to $x > 0$.

For more details on Lévy stable densities we refer the reader to specialized treatises, such as [2, 4, 9, 10, 13, 15], where different notations are adopted.

The required Mellin–Barnes integral[3] representation of the generic $L_\alpha^\theta(x)$ reads (see (6.5), (6.8), and (6.10) in [5])

$$L_\alpha^\theta(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha)\Gamma(1-s)}{\Gamma(\rho s)\Gamma(1-\rho s)} x^s\, ds, \quad 0 < \gamma < \min\{\alpha, 1\}, \tag{17}$$

where

$$\rho = \frac{\alpha - \theta}{2\alpha}. \tag{18}$$

In some cases, the integrand on the RHS of (17) can be simplified due to some cancelations. Noteworthy examples are provided by the Gaussian density $\{\alpha = 2, \theta = 0\}$, for which $\rho = 1/2$ and

$$L_2^0(x) = \frac{1}{2\sqrt{\pi}}\exp\left\{\frac{-x^2}{4}\right\} = \frac{1}{2x}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-s/2)} x^s\, ds, \quad 0 < \gamma < 1, \tag{19}$$

and by the unilateral (extremal) stable densities $\{0 < \alpha < 1, \theta = -\alpha\}$, for which $\rho = 1$ and

$$L_\alpha^{-\alpha}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha)}{\Gamma(s)} x^s\, ds, \quad 0 < \gamma < \alpha. \tag{20}$$

Thus, by virtue of the inversion formula (5) for the Mellin transform and the Mellin–Barnes representation (17), we can write for the generic strictly stable p.d.f.

$$L_\alpha^\theta(x) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha}\frac{\Gamma[-(s-1)/\alpha]\Gamma[1+(s-1)]}{\Gamma[1+\rho(s-1)]\Gamma[-\rho(s-1)]}, \tag{21}$$

where ρ is given by (18). Now it is possible to state the main theorem of this paper.

Theorem 3.1. *Let $L_{\alpha_p}^{\theta_p}(x;t)$, $L_{\alpha_q}^{\theta_q}(x;t)$, and $L_\beta^{\theta_\beta}(x;t)$ be strictly stable densities with exponents α_p, α_q, and β and asymmetry parameters θ_p, θ_q, and θ_β, respectively, such that $0 < \alpha_p \le 2$, $|\theta_p| \le \min\{\alpha_p, 2-\alpha_p\}$, $0 < \alpha_q \le 2$, $|\theta_q| \le \min\{\alpha_q, 2-\alpha_p\}$; $0 < \beta \le 1$, $|\theta_\beta| = \beta$. Then the following subordination representation holds true for $0 < x < \infty$:*

$$L_{\alpha_p}^{\theta_p}(x;t) = \int\limits_0^\infty L_{\alpha_q}^{\theta_q}(x;\tau) L_\beta^{-\beta}(\tau;t)\, d\tau, \quad \alpha_p = \beta\alpha_q, \quad \theta_p = \beta\theta_q. \tag{22}$$

Because of the self-similarity property of the stable p.d.f.'s, we can alternatively state

$$t^{-1/\alpha_p} L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right) = \int\limits_0^\infty \tau^{-1/\alpha_q} L_{\alpha_q}^{\theta_q}\left(\frac{x}{\tau^{1/\alpha_q}}\right) t^{-1/\beta} L_\beta^{-\beta}\left(\frac{\tau}{t^{1/\beta}}\right) d\tau. \tag{23}$$

The proof of (23) is a (straightforward) consequence of the previous considerations. By recalling the Mellin pairs for the involved stable densities (which can be easily obtained from (21) by adopting the correct parameters) and their scaling properties, we have

$$t^{-1/\alpha_p} L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right) \overset{\mathcal{M}}{\longleftrightarrow} t^{-1/\alpha_p}\left(\frac{1}{t^{1/\alpha_p}}\right)^{-s} \frac{1}{\alpha_p}\frac{\Gamma[-(s-1)/\alpha_p]\Gamma[1+(s-1)]}{\Gamma[1+\rho_p(s-1)]\Gamma[-\rho_p(s-1)]} \tag{24}$$

[3] The names refer to the two authors who, in the early 1910's, developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, these integrals were first used by S. Pincherle in 1888. For a revisited analysis of the pioneering work of Pincherle (1853-1936, Professor of Mathematics at the University of Bologna from 1880 to 1928), we refer the reader to the recent paper [6].

and

$$bx^a cL_\beta^{\theta_\beta}(cx^b) \overset{\mathcal{M}}{\longleftrightarrow} cc^{-(s+a)/b}\frac{1}{\beta}\frac{\Gamma[-(s+a)/(b\beta)+1/\beta]\Gamma[1+((s+a)/b-1)]}{\Gamma[1+\rho_\beta((s+a)/b-1)]\Gamma[-\rho_\beta((s+a)/b-1)]}. \tag{25}$$

After some algebra, we recognize

$$\mathcal{M}\left\{t^{-1/\alpha_p}L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right);s\right\} = \mathcal{M}\{bx^a cL_\beta^{\theta_\beta}(cx^b);s\}\mathcal{M}\{L_{\alpha_q}^{\theta_q}(x);s\} \tag{26}$$

provided that

$$\theta_\beta = -\beta, \qquad a = \alpha_q - 1, \qquad b = \alpha_q, \qquad c = t^{-1/\beta}, \tag{27}$$

and

$$\alpha_p = \beta\alpha_q, \qquad \theta_p = \beta\theta_q. \tag{28}$$

Recalling (7), we obtain from (26) the integral representation

$$t^{-1/\alpha_p}L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right) = \int_0^\infty \alpha_q\tau^{\alpha_q-1}L_{\alpha_q}^{\theta_q}\left(\frac{x}{\tau}\right)t^{-1/\beta}L_\beta^{-\beta}\left(\frac{\tau^{\alpha_q}}{t^{1/\beta}}\right)\frac{d\tau}{\tau}, \tag{29}$$

and, by making the replacement $\tau \to \tau^{1/\alpha_q}$, we finally get (23).

Taking into account the relationships in (28), we can point out some interesting subordination laws. In particular, we observe that any symmetric stable distribution with $\{0 < \alpha_p < 2, \theta_p = 0\}$ is subordinated to the Gaussian $\{\alpha_q = 2, \theta_q = 0\}$ and any stable distribution with $\{0 < \alpha_p < 1, |\theta_p| < \alpha_p\}$ is subordinated to the generalized Cauchy distribution $\{\alpha_q = 1, |\theta_q| < 1\}$, whose density is given in closed form as follows (see [5, (4.9)]):

$$L_1^{\theta_q}(x) = \frac{1}{\pi}\frac{\cos(\theta_q\pi/2)}{[x+\sin(\theta_q\pi/2)]^2+[\cos(\theta_q\pi/2)]^2}, \quad -\infty < x < +\infty. \tag{30}$$

4. Conclusions

We have given a nonstochastic derivation of a subordination formula for stochastic processes governed by strictly stable distributions by using the convolution rule of the Mellin transform. The results have been summarized in a theorem that provides the stability exponent and the asymmetry parameter of the subordinated process in terms of the corresponding parameters of the parent and directing processes.

It is known that the evolution in time of these stable distributions is governed by generalized diffusion equations containing partial derivatives of noninteger order in space, usually referred to as space fractional diffusion equations. In a forthcoming paper, this analysis will be extended to other self-similar stochastic processes with probability distributions governed by space-time fractional diffusion equations by using their Mellin–Barnes integral representations treated in [3, 5].

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