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The role of the Fox–Wright functions in fractional sub-diffusion of distributed order

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Abstract

The fundamental solution of the fractional diffusion equation of distributed order in time (usually adopted for modelling sub-diffusion processes) is obtained based on its Mellin–Barnes integral representation. Such solution is proved to be related via a Laplace-type integral to the Fox–Wright functions. A series expansion is also provided in order to point out the distribution of time-scales related to the distribution of the fractional orders. The results of the time fractional diffusion equation of a single order are also recalled and then re-obtained from the general theory.

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1. Introduction

The Wright function is defined by the series representation, valid in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k\Gamma(\lambda k+\mu)}, \quad \lambda > -1, \quad \mu \in \mathbf{C}, \; z \in \mathbf{C}. \tag{1.1}$$

It is an entire function of order $1/(1+\lambda)$, that has been known also as *generalized Bessel function*.[1]

Originally, Wright introduced and investigated this function with the restriction $\lambda \geqslant 0$ in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions [55–57]. Only later, in 1940, he considered the case $-1<\lambda<0$ [58]. We note that in the handbook of the Bateman Project see [15, vol. 3, Chapter 18], presumably for a misprint, λ is restricted to be non-negative in spite of the fact that the 1940 Wright's paper is cited.

For the cases $\lambda>0$ and $-1<\lambda<0$ we agree to distinguish the corresponding functions by calling them Wright functions of the first and second type, respectively. As a matter of fact the two types of functions exhibit a quite

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[1] When $\lambda = 1$ the Wright function can be expressed in terms of the Bessel function of order $\nu = \mu - 1$. In fact we have $J_{\mu-1}(z) = (z/2)^{\mu-1} W_{1,\mu}(-z^2/4)$.

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different asymptotic behaviour as it was shown more recently in two relevant papers by Wong and Zaho [53,54]. The case $\lambda = 0$ is trivial since it turns out from (1.1) $W_{0,\mu}(z) = \exp(z)/\Gamma(\mu)$.

Following a former idea of Wright himself [57], the Wright functions can be generalized as follows:

$$ {}_p\Psi_q(z) := \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_i + B_i k)} \frac{z^k}{k!}, \tag{1.2} $$

where $z \in \mathbf{C}$, $\{a_i, b_j\} \in \mathbf{C}$, $\{A_i, B_j\} \in \mathbf{R}$ with $A_i, B_j \neq 0$ and $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. An empty product, when it occurs, is taken to be 1.

The following alternative notations are commonly used:

$$ {}_p\Psi_q\left[\begin{matrix}(a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q}\end{matrix}; z\right] = {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}; z\right]. \tag{1.3} $$

Then, the standard Wright function (1.1), being obtained from (1.2) when $p = 0$ and $q = 1$ with $B_1 = \lambda > -1$, $b_1 = \mu$, reads

$$ W_{\lambda,\mu}(z) \equiv {}_0\Psi_1\left[\begin{matrix}-- \\ (\mu, \lambda)\end{matrix}; z\right]. \tag{1.4} $$

All the above functions are known to belong to the more general class of the Fox H functions introduced in 1961 by Fox [16]. For more information the interested reader is referred to the specialized literature including the books [25,28,41,45,50], and the relevant articles [23,24,26,51]. In particular, we recommend the article by Kilbas et al. [26] where the authors have established the conditions for the existence of ${}_p\Psi_q(z)$, see Section 2, and provided its representations in terms of Mellin–Barnes integrals, Section 3, and Fox H functions, Section 4.

For the sake of reader's convenience, we devote the Appendix for a short outline of the H functions in order to understand the Fox representation of the standard and generalized Wright functions of the first and second type that we shall introduce in the following. More appropriately, following [51,13], we can refer to the generalized Wright functions simply to as the Fox–Wright functions.

The Fox notation for the standard Wright functions depends on their type and reads

$$ W_{\lambda,\mu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\lambda k + \mu)} = \begin{cases} H^{1,0}_{0,2}\left[-z\left|\begin{matrix} - \;\; ; \;\; - \\ (0,1); (1-\mu, \lambda)\end{matrix}\right.\right], & \lambda > 0; \\ H^{1,0}_{1,1}\left[-z\left|\begin{matrix} -; (\mu, -\lambda) \\ (0,1); -\end{matrix}\right.\right], & -1 < \lambda < 0. \end{cases} \tag{1.5} $$

Putting $(b_1, B_1) = (\mu, \lambda)$, we have for the generalized Wright function:

$$ {}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + A_j k)}{\Gamma(\mu + \lambda k)\prod_{j=2}^{q} \Gamma(b_j + B_j k)} \frac{z^k}{k!}, \tag{1.6} $$

$$ {}_p\Psi_q(z) = \begin{cases} H^{1,p}_{p,q+1}\left[-z\left|\begin{matrix} (1-a_j, A_j)_{1,p} \;\; ; \;\; - \\ (0,1); (1-\mu, \lambda), (1-b_j, B_j)_{2,q}\end{matrix}\right.\right], & \lambda > 0; \\ H^{1,p}_{p+1,q}\left[-z\left|\begin{matrix} (1-a_j, A_j)_{1,p}; (\mu, -\lambda) \\ (0,1); (1-b_j, B_j)_{2,q}\end{matrix}\right.\right], & -1 < \lambda < 0. \end{cases} \tag{1.7} $$

In this paper we shall show the key-role of the standard and generalized Wright functions of the second type for finding the fundamental solutions of diffusion-like equations containing fractional derivatives in time of order $\beta < 1$. In the physical literature, such equations are in general referred to as *fractional sub-diffusion equations*, since they are used as model equations for the kinetic description of anomalous diffusion processes of slow type, characterized by a sub-linear growth of the variance (the mean squared displacement) with time. For an easy introduction to anomalous diffusion and fractional kinetics see the popular articles [29,49].

In addition to the simplest case of a single time-fractional derivative, more generally we can have a weighted (discrete or continuous) spectrum of time-fractional derivatives of distributed order (less than 1): then we speak about *fractional sub-diffusion of distributed order*. We note that only from a few years the fractional diffusion equations of distributed order have been investigated, over all to describe processes of *super-slow diffusion*. These processes are characterized by a variance growing as a power of the logarithm of time rather than as a linear combination of powers with exponent less than 1.

We shall devote Section 2 to the simplest case of the time-fractional diffusion equation of a single order: here we discuss how to obtain the fundamental solution that will be expressed in terms of a (standard) Wright function of second type. In Section 3 we shall consider the equations of distributed order. Starting from a generic distribution of fractional derivatives, we provide some representations of the fundamental solution involving Fox–Wright functions of the second type. In Section 4, as a check of consistency, we derive the fundamental solution for the single order as a particular case of the general representations. Finally, the main conclusions are drawn in Section 5.

2. The time-fractional diffusion equation of single order

It is well known that the fundamental solution (or *Green function*) of the standard diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbf{R}, \quad t \geqslant 0, \tag{2.1}$$

i.e. the solution subjected to the initial condition $u(x,0) = \delta(x)$ (the generalized Dirac function[2]), is the Gaussian *probability density function* (*pdf*)

$$u(x,t) = \frac{1}{2\sqrt{\pi}} t^{-1/2} \, \mathrm{e}^{-x^2/(4t)}, \tag{2.2}$$

that evolves in time with second moment[3] growing linearly with time,

$$\mu_2(t) := \int_{-\infty}^{+\infty} x^2 u(x,t) \, \mathrm{d}x = 2t. \tag{2.3}$$

We note the *scaling property* of the Green function, expressed by the equation

$$u(x,t) = t^{-1/2} \, U(x/t^{1/2}) \quad \text{with} \quad U(x) := u(x,1). \tag{2.4}$$

The function $U(x)$ depending on the single variable x turns out to be an even function of x, that is $U(x) = U(|x|)$, and is called the *reduced Green function*. The positive variable $X := |x|/t^{1/2}$ is known as the similarity variable.

By replacing in the standard diffusion equation (2.1) the first-order time derivative by an *integro-differential* operator interpreted as a time fractional derivative of order $\beta \in (0, 1]$, we obtain a generalized diffusion equation, the parabolic character of which is preserved. We call it the *time-fractional diffusion equation of order* β and, consistently with (2.1), we write it as

$$\frac{\partial^\beta}{\partial t^\beta} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbf{R}, \quad t \geqslant 0, \ 0 < \beta \leqslant 1, \tag{2.5}$$

with $u(x,0) = \delta(x)$.

In (2.5) $\partial^\beta/\partial t^\beta$ denotes the fractional derivative (of Caputo type) of order β, whose definition is more easily understood if given in terms of Laplace transform. Let $f(t)$ be a sufficiently well-behaved (generalized) function on $t \geqslant 0$ with Laplace transform $\mathrm{L}\{f(t); s\} = \widetilde{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t) \, \mathrm{d}t$. We have

$$L\left\{\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t); s\right\} = s^\beta \widetilde{f}(s) - s^{\beta-1} f(0^+) \quad \text{with } 0 < \beta \leqslant 1, \tag{2.6}$$

[2] REMARK: Through this paper we are working *formally* in that we assume a suitable space of generalized functions where it is possible to deal at the same time with delta functions, integral transforms of Fourier, Laplace, Mellin type, and fractional integrals and derivatives.

[3] The centred second moment provides the variance usually denoted by $\sigma^2(t)$. It is a measure for the spatial spread of $u(x,t)$ with time of a random walking particle starting at the origin $x = 0$, pertinent to the solution of the diffusion equation (2.1) with initial condition $u(x,0) = \delta(x)$. The asymptotic behaviour of the variance as $t \to \infty$ is relevant to distinguish *normal diffusion* ($\sigma^2(t)/t \to c > 0$) from anomalous processes of *sub-diffusion* ($\sigma^2(t)/t \to 0$) and of *super-diffusion* ($\sigma^2(t)/t \to +\infty$).

if we define

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{(t-\tau)^\beta} & \text{for } 0<\beta<1, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} f(t) & \text{for } \beta=1. \end{cases} \tag{2.7}$$

For $0<\beta<1$ we can also write the fractional derivative (2.7) in each of the following two forms,

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_0^t \frac{f(\tau)-f(0^+)}{(t-\tau)^\beta} \mathrm{d}\tau \right], \tag{2.8}$$

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_0^t \frac{f(\tau)}{(t-\tau)^\beta} \mathrm{d}\tau \right] - f(0^+) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{2.9}$$

We refer to the fractional derivative defined by (2.7) as the *Caputo* fractional derivative, since it was formerly applied by Caputo in the late sixties for modelling dissipation effects in *Linear Viscoelasticity*, see e.g. [5,6,9,33]. The reader should observe that Caputo's definition differs from the usual one named after Riemann and Liouville, which is given by the first term in the RHS of (2.7), see e.g. [4,46]. For more details we refer e.g. to [21,27,44].

Returning to Eq. (2.5), its fundamental solution can be obtained by applying in sequence the Fourier and Laplace transforms to the equation itself.[4]

Let $f(x)$ be a sufficiently well-behaved (generalized) function on $x\in\mathbf{R}$ with Fourier transform $\mathrm{F}\{f(x);\kappa\}=\hat{f}(\kappa)=\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\kappa x} f(x)\,\mathrm{d}x,\ \ \kappa\in\mathbf{R}$. We have

$$F\left\{ \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x);\kappa \right\} = -\kappa^2 \widehat{f}(\kappa) \tag{2.10}$$

and for the Dirac generalized function $\delta(x)$ we have $\widehat{\delta}(\kappa)\equiv 1$. Then, in the Fourier–Laplace domain our Cauchy problem (2.3) appears, after applying the formulas (2.6), (2.10), in the form $s^\beta\widehat{\widetilde{u}}(\kappa,s)-s^{\beta-1}=-\kappa^2\widehat{\widetilde{u}}(\kappa,s)$, from which we obtain

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta+\kappa^2}, \quad 0<\beta\leqslant 1,\ \ \Re(s)>0,\ \ \kappa\in\mathbf{R}. \tag{2.11}$$

To determine the Green function (that is expected to be symmetric in x) in the space–time domain we can follow two alternative strategies related to the different order in carrying out the inversion of the Fourier–Laplace transforms in (2.11).

(S1) invert the Fourier transform getting $\widetilde{u}(x,s)$ and then invert this Laplace transform;
(S2) invert the Laplace transform getting $\widehat{u}(\kappa,t)$ and then invert this Fourier transform.

Strategy (*S*1): Recalling the Fourier transform pair,

$$\frac{a}{b+\kappa^2} \overset{\mathscr{F}}{\leftrightarrow} \frac{a}{2b^{1/2}} \mathrm{e}^{-|x|b^{1/2}}, \quad b>0, \tag{2.12}$$

[4] The time-fractional diffusion equation was investigated by using Mellin transforms by Schneider and Wyss [47] in their pioneering 1989 paper where they adopted the equivalent integral form

$$u(x,t)=u(x,0)+\frac{1}{\Gamma(\beta)}\int_0^t \left[\frac{\partial^2}{\partial x^2}u(x,\tau)\right]\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}.$$

The time-fractional diffusion equation with the Caputo derivative has been adopted and investigated by several authors. From the former contributors let us quote Mainardi, see e.g. [30–33] (see also [19,20,22,36] and references therein), who has expressed the fundamental solution in terms of a special function (of Wright type) of which he has studied the analytical properties and provided plots also for $1<\beta<2$.

and setting $a = s^{\beta-1},\ b = s^{\beta}$ we get

$$\widetilde{u}(x,s) = \frac{s^{\beta/2-1}}{2}\,\mathrm{e}^{-|x|s^{\beta/2}}, \quad 0 < \beta \leqslant 1. \tag{2.13}$$

The strategy (S1) has been followed by Mainardi [30–33] to obtain the Green function in the form

$$u(x,t) = t^{-\beta/2}\,U(|x|/t^{\beta/2}), \quad -\infty < x < +\infty,\ t \geqslant 0, \tag{2.14}$$

where the variable $X := |x|/t^{\beta/2}$ acts as *similarity variable* and the function $U(x) := u(x,1)$ denotes the *reduced Green function* that is expressed in terms of a Wright function of the second type. Indeed we have

$$U(x) = \tfrac{1}{2}\,M_{\beta/2}(|x|) = \tfrac{1}{2}\,W_{-\beta/2,1-\beta/2}(-|x|), \tag{2.15}$$

where the M function of order $\beta/2$ has been introduced and investigated in [30–33], see also [44]. More generally, in the complex plain the function $M_{\beta/2}(z)$ is well defined for any $\beta \in (0,2)$ and $\forall z \in \mathbf{C}$ by a power series as

$$\begin{aligned} M_{\frac{\beta}{2}}(z) &= \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma[-\beta k/2 + (1-\beta/2)]} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-z)^k}{k!}\,\Gamma[(\beta(k+1))/2]\,\sin[(\pi\beta(k+1)/2]. \end{aligned} \tag{2.16}$$

By comparing the power series in (1.1) and (2.16) we recognize that the $M_{\beta/2}$ function is indeed a special case of the Wright function of the second type with $\lambda = -\beta/2$ and $\mu = 1-\beta/2$, so that it is an entire function of order $1/(1-\beta/2)$. Noteworthy special cases of this function are

$$M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}}\,\exp(-z^2/4), \quad M_{1/3}(z) = 3^{2/3}\,\mathrm{Ai}(z/3^{1/3}), \tag{2.17}$$

where Ai denotes the *Airy function*, see e.g. [1,52].

Strategy (*S*2): Recalling the Laplace transform pair, see e.g. [15,21,44],

$$\frac{s^{\beta-1}}{s^{\beta}+c} \overset{\mathcal{L}}{\leftrightarrow} E_{\beta}(-ct^{\beta}), \quad c > 0, \tag{2.18}$$

and setting $c = \kappa^2$ we get

$$\widehat{u}(\kappa,t) = E_{\beta}(-\kappa^2 t^{\beta}), \quad 0 < \beta \leqslant 1, \tag{2.19}$$

where E_{β} denotes the Mittag–Leffler function.[5]

[5] Let us recall that the Mittag–Leffler function $E_{\beta}(z)$ $(\beta > 0)$ is an entire transcendental function of order $1/\beta$, defined in the complex plane by the power series

$$E_{\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}, \quad \beta > 0,\ z \in \mathbf{C}.$$

Originally Mittag–Leffler introduced and investigated (in five notes from 1903 to 1905) this function as an instructive example of entire function that generalizes the exponential (recovered for $\beta = 1$). For more details we refer e.g. to [14,15,18,34,44]. Here we like to recall that, for $0 < \beta < 1$ and negative argument, E_{β} preserves the *complete monotonicity* of the exponential: indeed it is represented in terms of a real Laplace transform of a positive function,

$$E_{\beta}(-t^{\beta}) = \frac{\sin(\beta\pi)}{\pi}\int_0^{\infty} \mathrm{e}^{-\sigma t}\,\frac{\sigma^{\beta-1}}{\sigma^{2\beta} + 2\sigma^{\beta}\cos(\beta\pi) + 1}\,\mathrm{d}\sigma, \quad t \geqslant 0,\ 0 < \beta < 1,$$

but decreases at infinity as a power law with exponent $-\beta$: $E_{\beta}(-t^{\beta}) \sim t^{-\beta}/\Gamma(-\beta)$. In particular, if $\beta = \frac{1}{2}$ we have, for $t \geqslant 0$ and $t \to \infty$,

$$E_{1/2}(-\sqrt{t}) = \mathrm{e}^{t}\,\mathrm{erfc}(\sqrt{t}) \sim 1/(\sqrt{\pi t}),$$

where erfc denotes the *complementary error* function, see e.g. [1,52].

The strategy (S2) has been followed by Gorenflo et al. [17] and by Mainardi et al. [35] to obtain the Green functions of the more general space–time fractional diffusion equations in terms of Mellin–Barnes integrals. For the time fractional diffusion equation the reduced Green function (2.15) now appears in the form:

$$U(x) = \frac{1}{\pi}\int_0^{\infty} \cos(\kappa x) E_\beta(-\kappa^2)\,\mathrm{d}\kappa = \frac{1}{2x}\,\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{\Gamma(1-s)}{\Gamma(1-\beta s/2)}\,x^s\,\mathrm{d}s \tag{2.20}$$

with $0<\gamma<1$ and $x>0$. We point out that from now on we restrict our attention to $x>0$ in view of the symmetry of the solution.

In conclusion, we may represent the solution $U(x)$ given in (2.15) and in (2.20) using the general formalism of the Fox–Wright functions, that is in terms of a generalized Wright function ${}_p\Psi_q$ [17], or in terms of a Fox H function [38], as follows

$$U(x) = \frac{1}{2}\,{}_0\Psi_1\left[\begin{matrix} -- \\ (1-\frac{\beta}{2}, -\frac{\beta}{2}) \end{matrix}; -x\right] = \frac{1}{2}\,H^{1,0}_{1,1}\left[x\,\left|\begin{matrix} --; (1-\frac{\beta}{2}, \frac{\beta}{2}) \\ (0,1); -- \end{matrix}\right.\right]. \tag{2.21}$$

As proven in [35] we recall that $u(x,t)$ can interpreted as a symmetric spatial *pdf* evolving in time, with a stretched exponential decay. More precisely, we have

$$U(x) = \tfrac{1}{2}\,M_{\frac{\beta}{2}}(|x|) \sim A x^a \mathrm{e}^{-bx^c}, \quad x\to+\infty, \tag{2.22}$$

with

$$A = \{2\pi(2-\beta)\,2^{\beta/(2-\beta)}\beta^{(2-2\beta)/(2-\beta)}\}^{-1/2}, \tag{2.23}$$

$$a = \frac{2\beta-2}{2(2-\beta)}, \quad b=(2-\beta)2^{-2/(2-\beta)}\beta^{\beta/(2-\beta)}, \quad c=\frac{2}{2-\beta}. \tag{2.24}$$

Furthermore the moments (of even order) of $u(x,t)$ are

$$\mu_{2n}(t) := \int_{-\infty}^{+\infty} x^{2n}\,u(x,t)\,\mathrm{d}x = \frac{\Gamma(2n+1)}{\Gamma(\beta n+1)}t^{\beta n}, \quad n=0,1,2,\ldots, \quad t\geqslant 0. \tag{2.25}$$

Of particular interest is the evolution of the second moment: from (2.25) we have

$$\mu_2(t) = 2\,\frac{t^\beta}{\Gamma(\beta+1)}, \quad 0<\beta\leqslant 1, \tag{2.26}$$

so that for $\beta<1$ we note a sub-linear growing in time, consistently with an anomalous process of *slow diffusion* (alternatively called *sub-diffusion*), in contrast with the law (2.3) of normal diffusion. Such result can also be obtained in a simpler way from the Fourier transform (2.19) noting that

$$\mu_2(t) = -\frac{\partial^2}{\partial\kappa^2}\,\widehat{u}(\kappa=0,t). \tag{2.27}$$

3. The time-fractional diffusion equation of distributed order

The fractional diffusion equation (2.5) can be generalized by using the notion of fractional derivative of distributed order in time.[6] We now consider the so-called *time-fractional diffusion equation of distributed order*

$$\int_0^1 b(\beta)\left[\frac{\partial^\beta}{\partial t^\beta}\,u(x,t)\right]\mathrm{d}\beta = \frac{\partial^2}{\partial x^2}\,u(x,t), \quad b(\beta)\geqslant 0, \quad \int_0^1 b(\beta)\,\mathrm{d}\beta = 1, \tag{3.1}$$

with $x\in\mathbf{R}$, $t\geqslant 0$. Clearly, some special conditions of regularity and behaviour near the boundaries will be required for the weight function $b(\beta)$.

[6] We find a former idea of fractional derivative of distributed order in time in the 1969 book by Caputo [6], that was later developed by Caputo himself, see [7,8] and by Bagley and Torvik, see [2].

Time-fractional diffusion equations of distributed order have recently been discussed in [10–12,48] and in [42]. As usual we consider the initial condition $u(x,0)=\delta(x)$ in order to keep the probability meaning. Indeed, already in the paper [10] it was shown that the Green function is non-negative and normalized, so allowing interpretation as a density of the probability at time t of a diffusing particle to be in the point x. The main interest of those authors was devoted to the second moment of the Green function (the displacement variance or mean-square displacement) in order to show the sub-diffusive character of the related stochastic process by analysing some interesting cases of the weight function $b(\beta)$.

In this paper we are interested to a more general approach involving a generic distribution $b(\beta)$ in order to provide a general representation of the corresponding fundamental solution. By applying in sequence the Fourier and Laplace transforms to Eq. (3.1) in analogy with the single-order case, see Eqs. (2.5) and (2.11), we obtain,

$$\widehat{\widetilde{u}}(\kappa,s)=\frac{B(s)/s}{B(s)+\kappa^2},\quad \Re(s)>0,\quad \kappa\in\mathbf{R}, \tag{3.2}$$

where

$$B(s)=\int_0^1 b(\beta)\,s^\beta\,\mathrm{d}\beta. \tag{3.3}$$

Before of trying to get the solution in the space–time domain, it is worth to outline the expression of its second moment as it can be derived from Eq. (3.2) using (2.27). We have

$$\widehat{\widetilde{u}}(\kappa,s)=\frac{1}{s}\left(1-\frac{\kappa^2}{B(s)}+\cdots\right)\quad \text{so } \widetilde{\mu_2}(s)=-\frac{\partial^2}{\partial\kappa^2}\widehat{\widetilde{u}}(\kappa=0,s)=\frac{2}{s\,B(s)}. \tag{3.4}$$

Then, from (3.4) we are allowed to derive the asymptotic behaviours of $\mu_2(t)$ for $t\to 0^+$ and $t\to+\infty$ from the asymptotic behaviours of $B(s)$ for $s\to\infty$ and $s\to 0$, respectively, in virtue of the Tauberian theorems.

The expected sub-linear growth with time is shown in the following special cases of $b(\beta)$ treated in [10,11].

The first case is *slow diffusion* (power-law growth) where

$$b(\beta)=b_1\delta(\beta-\beta_1)+b_2\delta(\beta-\beta_2),\quad 0<\beta_1<\beta_2\leqslant 1,\ \ b_1>0,\ \ b_2>0,\ \ b_1+b_2=1.$$

In fact

$$\widetilde{\mu_2}(s)=\frac{2}{b_1\,s^{\beta_1+1}+b_2\,s^{\beta_2+1}}\quad \text{so}\ \ \mu_2(t)\sim\begin{cases}\dfrac{2}{b_2\Gamma(\beta_2+1)}\,t^{\beta_2}, & t\to 0,\\[2mm] \dfrac{2}{b_1\Gamma(\beta_1+1)}\,t^{\beta_1}, & t\to\infty.\end{cases} \tag{3.5}$$

In [10], see Eq. (16), the authors were able to provide the analytical expression of $\mu_2(t)$ in terms of a 2-parameter Mittag–Leffler function.

The second case is *super-slow diffusion* (logarithmic growth) where

$$b(\beta)=1,\quad 0\leqslant\beta\leqslant 1.$$

In fact

$$\widetilde{\mu_2}(s)=2\,\frac{\log s}{s(s-1)}\quad \text{so}\ \ \mu_2(t)\sim\begin{cases}2t\ \log(1/t), & t\to 0,\\ 2\ \log(t), & t\to\infty.\end{cases} \tag{3.6}$$

In [10], see Eqs. (23)–(26), the authors were able to provide the analytical expression of $\mu_2(t)$ in terms of an exponential integral function.

Let us now return to Eq. (3.2). Inverting the Laplace transform, in virtue of the Titchmarsh theorem we obtain

$$\widehat{u}(\kappa,t)=-\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{Im}\,\{\widehat{\widetilde{u}}(r\mathrm{e}^{\mathrm{i}\pi})\}\,\mathrm{d}r, \tag{3.7}$$

that requires the expression of $-\mathrm{Im}\{B(s)/[s(B(s)+\kappa^2)]\}$ along the ray $s = r\,\mathrm{e}^{\mathrm{i}\pi}$ with $r > 0$ (the branch cut of the functions s^β and $s^{\beta-1}$). By writing

$$B(r\,\mathrm{e}^{\mathrm{i}\pi}) = \rho\cos(\pi\gamma) + \mathrm{i}\rho\sin(\pi\gamma), \qquad \begin{cases} \rho = \rho(r) = |B(r\mathrm{e}^{\mathrm{i}\pi})|, \\ \gamma = \gamma(r) = \dfrac{1}{\pi}\,\arg\,[B(r\,\mathrm{e}^{\mathrm{i}\pi})], \end{cases} \tag{3.8}$$

after simple calculations we get

$$\widehat{u}(\kappa, t) = \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\, K(\kappa, r)\,\mathrm{d}r, \tag{3.9}$$

where

$$K(\kappa, r) = \frac{1}{\pi}\,\frac{\kappa^2\rho\,\sin(\pi\gamma)}{\kappa^4 + 2\kappa^2\,\rho\cos(\pi\gamma) + \rho^2}. \tag{3.10}$$

Then, since $u(x,t)$ is symmetric in x, the inversion formula for the Fourier transform yields for $x, t \geqslant 0$,

$$u(x,t) = \frac{1}{\pi}\int_0^{+\infty} \cos(\kappa x)\,\left\{\int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\; K(r,\kappa)\,\mathrm{d}r\right\}\,\mathrm{d}\kappa. \tag{3.11}$$

To carry out the above Fourier integral we use the method of the Mellin transform. Let

$$\mathscr{M}\{f(\xi); s\} = f^*(s) = \int_0^{+\infty} f(\xi)\xi^{s-1}\,\mathrm{d}\xi, \quad \gamma_1 < \Re(s) < \gamma_2, \tag{3.12}$$

be the Mellin transform of a sufficiently well-behaved function $f(\xi)$, and let

$$\mathscr{M}^{-1}\{f^*(s);\xi\} = f(\xi) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} f^*(s)\xi^{-s}\,\mathrm{d}s, \tag{3.13}$$

be the inverse Mellin transform, where $\xi > 0$, $\gamma = \Re(s)$, $\gamma_1 < \gamma < \gamma_2$. Denoting by $\overset{\mathscr{M}}{\leftrightarrow}$ the juxtaposition of a function $f(\xi)$ with its Mellin transform $f^*(s)$, the Mellin convolution implies

$$h(\xi) = f(\xi) \otimes g(\xi) := \int_0^\infty \frac{1}{\eta}\, f(\eta)\, g(\xi/\eta)\,\mathrm{d}\eta \overset{\mathscr{M}}{\leftrightarrow} h^*(s) = f^*(s)\, g^*(s). \tag{3.14}$$

Then, following [35] (pp. 160–161), we recognize that the Fourier integral in (3.11) can be interpreted as a Mellin convolution in κ, that is $u(x,t) = f(\kappa,t) \otimes g(\kappa,x)$, if we set (see (3.14) with $\xi = 1/x$, $\eta = \kappa$)

$$f(\kappa,t) := \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\, K(\kappa,r)\,\mathrm{d}r \overset{\mathscr{M}}{\leftrightarrow} f^*(s,t), \tag{3.15}$$

$$g(\kappa,x) := \frac{1}{\pi\, x\, \kappa}\cos\left(\frac{1}{\kappa}\right) \overset{\mathscr{M}}{\leftrightarrow} \frac{\Gamma(1-s)}{\pi\, x}\sin\left(\frac{\pi s}{2}\right) := g^*(s,x), \tag{3.16}$$

with $0 < \Re(s) < 1$. The next step thus consists in computing the Mellin transform $f^*(s,t)$ of the function $f(\kappa,t)$ and then inverting the product $f^*(s,t)\,g^*(s,x)$ using (3.16) in the inversion Mellin formula, namely

$$\begin{aligned} u(x,t) &= \frac{1}{\pi x}\,\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} f^*(s,t)\Gamma(1-s)\sin(\pi s/2)\,x^s\,\mathrm{d}s \\ &= \frac{1}{x}\,\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} f^*(s,t)\,\frac{\Gamma(1-s)}{\Gamma(s/2)\Gamma(1-s/2)}\,x^s\,\mathrm{d}s. \end{aligned} \tag{3.17}$$

The required Mellin transform $f^*(s;t)$ is

$$f^*(s,t) = \int_0^\infty \frac{\mathrm{e}^{-rt}}{r} \left\{ \frac{1}{\pi} \int_0^\infty \frac{\kappa^2 \rho \sin(\pi\gamma)}{\kappa^4 + 2\rho\cos(\pi\gamma)\kappa^2 + \rho^2} \kappa^{s-1} \, \mathrm{d}\kappa \right\} \mathrm{d}r. \tag{3.18}$$

The term in braces can be computed by making the variable change $\kappa^2 \to \rho\mu$ and reads

$$\frac{\rho^{s/2+1}}{2\rho} \frac{1}{\pi} \int_0^\infty \frac{\sin(\pi\gamma)}{\mu^2 + 2\mu\cos(\pi\gamma) + 1} \mu^{(s/2+1)-1} \, \mathrm{d}\mu = -\frac{\rho^{s/2}}{2} \left\{ \frac{\Gamma(s/2+1)\,\Gamma[1-(s/2+1)]}{\Gamma(\gamma s/2)\,\Gamma(1-\gamma s/2)} \right\}, \tag{3.19}$$

where we have used a formula of the Handbook by Marichev, see [39, p. 156, Eq. (15) (1)], under the condition $0 < \Re(s/2+1) < 2$, $|\gamma| < 1$. As a consequence of (3.18)–(3.19) we finally get

$$f^*(s,t) = -\int_0^\infty \frac{\mathrm{e}^{-rt}}{r} \frac{\rho^{s/2}}{2} \left\{ \frac{\Gamma(s/2+1)\,\Gamma[1-(s/2+1)]}{\Gamma(\gamma s/2)\,\Gamma(1-\gamma s/2)} \right\} \mathrm{d}r, \tag{3.20}$$

Now, using Eqs. (3.17) and (3.20) we can finally write the solution as

$$u(x,t) = \frac{1}{2\pi x} \int_0^\infty \frac{\mathrm{e}^{-rt}}{r} F(\rho^{1/2}x) \, \mathrm{d}r, \tag{3.21}$$

where $F(\rho^{1/2}x)$ is expressed in terms of Mellin–Barnes integrals:

$$\begin{aligned} F(\rho^{1/2}x) &= \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{\pi\Gamma(1-s)}{\Gamma(\gamma s/2)\Gamma(1-\gamma s/2)} (\rho^{1/2}x)^s \, \mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \Gamma(1-s) \sin(\pi\gamma s/2) (\rho^{1/2}x)^s \, \mathrm{d}s, \end{aligned} \tag{3.22}$$

and $\rho = \rho(r)$, $\gamma = \gamma(r)$, we remind it, are related to the distribution $b(\beta)$ according to Eqs. (3.3) and (3.8). By solving the Mellin–Barnes integrals by the residue theorem we arrive at the series representations in powers of $(\rho^{1/2}x)$,

$$\begin{aligned} F(\rho^{1/2}x) &= \pi\rho^{1/2}x \sum_{k=0}^\infty \frac{(-\rho^{1/2}x)^k}{k!\Gamma(\gamma k/2 + \gamma/2)\Gamma(-\gamma k/2 + 1 - \gamma/2)} \\ &= \rho^{1/2}x \sum_{k=0}^\infty \frac{(-\rho^{1/2}x)^k}{k!} \sin(\pi\gamma(k+1)/2). \end{aligned} \tag{3.23}$$

Then, in virtue of Eqs. (1.2)–(1.3) we recognize

$$F(\rho^{1/2}x) = \pi\rho^{1/2}x \, {}_0\Psi_2 \left[\begin{matrix} -- \\ (1-\gamma/2, -\gamma/2)(\gamma/2, \gamma/2) \end{matrix} ; -\rho^{1/2}x \right], \tag{3.24}$$

which implies that $F(\rho^{1/2}x)$ is a *Fox–Wright function of the second type* (being $\gamma > 0$). The Fox representation of the function is

$$F(\rho^{1/2}x) = \pi H_{1,2}^{1,0} \left[\rho^{1/2}x \left| \begin{matrix} --; (1, \gamma/2) \\ (1,1); (1, \gamma/2) \end{matrix} \right. \right]. \tag{3.25}$$

In conclusion, the fundamental solution admits the (equivalent) representations:

$$u(x,t) = \frac{1}{2} \int_0^\infty \frac{\mathrm{e}^{-rt}}{r} \rho^{1/2} \, {}_0\Psi_2 \left[\begin{matrix} -- \\ (1-\gamma/2, -\gamma/2)(\gamma/2, \gamma/2) \end{matrix} ; -\rho^{1/2}x \right] \mathrm{d}r \tag{3.26}$$

and

$$u(x,t) = \frac{1}{2x}\int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\, H^{1,0}_{1,2}\left[\rho^{1/2}x \,\middle|\, \begin{matrix} --;\,(1,\gamma/2) \\ (1,1);\,(1,\gamma/2) \end{matrix}\right] \mathrm{d}r. \tag{3.27}$$

If we exchange the order of integration and summation, we have an alternative and interesting series representation of the fundamental solution:

$$u(x,t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\,\varphi_k(t), \tag{3.28}$$

where

$$\varphi_k(t) = \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\,\sin[\pi\gamma(k+1)/2]\,\rho^{(k+1)/2}\,\mathrm{d}r, \tag{3.29}$$

with $\rho=\rho(r)$, $\gamma=\gamma(r)$.

4. The reduction to fractional sub-diffusion of a single order v

In order to check the consistency of the general analysis carried out in the previous section and to explore directions for future work, we find it instructive to derive as particular cases the results of Section 2 concerning the fractional sub-diffusion of a single order. We now agree to denote this (fixed) order by v to be distinguishes from β used in the distributed order case (as a variable order). This means to consider in Eq. (3.1) the particular case

$$b(\beta) = \delta(\beta - v), \quad 0 < v < 1, \tag{4.1}$$

so that $B(s) = s^v$ and Eq. (3.8) yields

$$\rho = \rho(r) = r^v, \quad \gamma = \text{const.} = v. \tag{4.2}$$

In this case the Eqs. (3.9)–(3.10) reduce to

$$\widehat{u}(\kappa,t) = \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\,K(\kappa,r)\,\mathrm{d}r, \quad K(\kappa,r) = \frac{1}{\pi}\,\frac{\kappa^2\rho\,\sin(\pi v)}{\kappa^4 + 2\kappa^2\rho\,\cos(\pi v) + \rho^2}, \tag{4.3}$$

and hence, in virtue of (3.11),

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \cos(kx)\left\{\frac{\sin(\pi v)}{\pi}\int_0^\infty \frac{\kappa^2 r^{v-1}\,\mathrm{e}^{-rt}}{\kappa^4 + 2r^v\cos(\pi v)\kappa^2 + r^{2v}}\,\mathrm{d}r\right\}\mathrm{d}\kappa. \tag{4.4}$$

With the change of variable $r = \sigma\kappa^{2/v}$ the term in brace reads

$$\frac{\sin(v\pi)}{\pi}\int_0^\infty \mathrm{e}^{-\sigma\kappa^{2/v}t}\,\frac{\sigma^{v-1}}{\sigma^{2v} + 2\sigma^v\cos(v\pi) + 1}\,\mathrm{d}\sigma, \tag{4.5}$$

so that

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \cos(kx)\,E_v(-\kappa^2 t^v)\,\mathrm{d}\kappa, \tag{4.6}$$

where E_v is the Mittag–Leffler function of order v according to its integral representation of footnote 5. We thus recognize that Eq. (4.6) is consistent with Eq. (2.19), and, once applied the scaling relation for the Fourier transform, with Eqs. (2.14)–(2.15) and (2.20).

The consistency with the results expressed in terms of the general formalism of Fox–Wright function, that is the comparison between Eqs. (2.21) and (3.26)–(3.27), can be obtained in a less direct way because one is required to use the scaling relations and the Laplace transform rules of the Fox H functions available in the specialized literature. We do not report on this tedious calculations. In a more direct and instructive way the consistency with the single-order case is shown by using the series representation of the fundamental solution (3.28)–(3.29). In this special case the functions $\varphi_k(t)$ turn out to be

$$\begin{aligned}\varphi_k(t) &= \sin[\pi\nu(k+1)/2] \int_0^\infty \frac{\mathrm{e}^{-rt}}{r}\, r^{\nu(k+1)/2}\,\mathrm{d}r \\ &= \sin[\pi\nu(k+1)/2]\, \frac{\Gamma[\nu(k+1)/2]}{t^{\nu(k+1)/2}}.\end{aligned} \tag{4.7}$$

As a consequence, the solution reads

$$\begin{aligned}u(x,t) &= \frac{1}{2}\, t^{-\nu/2} \cdot \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-x/t^{\nu/2})^k}{k!}\, \Gamma[\nu(k+1)/2] \sin[\pi\nu(k+1)/2] \\ &= \frac{1}{2}\, t^{-\nu/2} M_{\frac{\nu}{2}}\left(\frac{x}{t^{\nu/2}}\right),\end{aligned} \tag{4.8}$$

in agreement with Eqs. (2.14)–(2.16). Of course, only in this special case it is possible to single out a common time factor $(t^{-\nu/2})$ from all the functions $\varphi_k(t)$ and get a self-similar solution. In general the set of functions $\varphi_k(t)$ give raise to a distribution of different time scales related in some way to the distribution of the orders of the fractional derivatives.

5. Conclusions

The diffusion-like equations containing fractional derivatives in time and/or in space are usually adopted to model phenomena of anomalous transport in physics, so a detailed study of their solutions is required. Our attention in this paper has been focused on the time fractional diffusion equations of distributed order less than 1, which are known to be model equations for sub-diffusive processes. Specifically, we have worked out how express their fundamental solutions in terms of Fox–Wright functions.

At first we have recalled the main results for the fundamental solution of the time fractional diffusion equation of a single order, which are obtained by applying the Fourier–Laplace integral transforms. The required solution turns out to be self similar (through a definite space–time scaling relationship), and expressed in terms of a special function belonging to the simpler class of the Wright functions. Then we were able to adapt the previous techniques for obtaining the fundamental solution in the general case of a distributed order. For such solution we have provided a representation in terms of a Laplace-type integral of a Fox–Wright function, that can be expanded in a series containing powers of space and certain functions of time, responsible of the time-scale distribution.

Among the various questions for future research on this topic, particularly relevant in our opinion is the possibility to use our analytical results for plotting the fundamental solutions in some noteworthy cases of fractional order distribution, as it was for the simplest case of a single order.

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Appendix A. The Fox H functions

According to a standard notation the Fox H function is defined as

$$H^{m,n}_{p,q}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathscr{L}} \mathscr{H}^{m,n}_{p,q}(s) z^s\, \mathrm{d}s, \tag{A.1}$$

where $\mathscr{L}$ is a suitable path in the complex plane $\mathbf{C}$ to be disposed later, $z^s = \exp\{s(\log|z| + \mathrm{i}\,\arg z)\}$, and

$$\mathscr{H}^{m,n}_{p,q}(s) = \frac{A(s)B(s)}{C(s)D(s)}, \tag{A.2}$$

$$A(s) = \prod_{j=1}^{m} \Gamma(b_j - B_j s), \quad B(s) = \prod_{j=1}^{n} \Gamma(1 - a_j + A_j s), \tag{A.3}$$

$$C(s) = \prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j s), \quad D(s) = \prod_{j=n+1}^{p} \Gamma(a_j - A_j s) \tag{A.4}$$

with $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q$,$\{a_j, b_j\} \in \mathbf{C}$, $\{A_j, B_j\} \in \mathbf{R}^+$. An empty product, when it occurs, is taken to be one so

$$n = 0 \Longleftrightarrow B(s) = 1, \quad m = q \Longleftrightarrow C(s) = 1, \quad n = p \Longleftrightarrow D(s) = 1.$$

Due to the occurrence of the factor z^s in the integrand of (A.1), the H function is, in general, multi-valued, but it can be made one-valued on the Riemann surface of $\log z$ by choosing a proper branch. We also note that when the A's and B 's are equal to 1, we obtain the Meijer's G-functions $G^{m,n}_{p,q}(z)$.

The above integral representation of the H functions, by involving products and ratios of Gamma functions, is known to be of *Mellin–Barnes integral* type.[7] A compact notation is usually adopted for (A.1):

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left[z \left| \begin{matrix} (a_j, A_j)_{j=1,n}; (a_j, A_j)_{j=n+1,p} \\ (b_j, B_j)_{j=1,m}; (b_j, B_j)_{j=m+1,q} \end{matrix} \right. \right]. \tag{A.5}$$

Thus, the singular points of the kernel $\mathscr{H}$ are the poles of the Gamma functions entering the expressions of $A(s)$ and $B(s)$, that we assume do not coincide. Denoting by $\mathscr{P}(A)$ and $\mathscr{P}(B)$ the sets of these poles, we write $\mathscr{P}(A) \cap \mathscr{P}(B) = \emptyset$. The conditions for the existence of the H-functions can be made by inspecting the convergence of the integral (A.1), which can depend on the selection of the contour $\mathscr{L}$ and on certain relations between the parameters $\{a_i, A_i\}$ $(i = 1, \ldots, p)$ and $\{b_j, B_j\}$ $(j = 1, \ldots, q)$. For the analysis of the general case we refer to the specialized treatises on H functions, e.g. [40,41,50] and, in particular to the paper by Braaksma [3], where an exhaustive discussion on the asymptotic expansions and analytical continuation of these functions is found, see also [24].

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[7] As historical note we point out that the names refer to the two authors, who in the first 1910s developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, these integrals were first used in 1888 by Pincherle, see e.g. [37]. Recent treatises on Mellin–Barnes integrals are those by Marichev [39] and Paris and Kaminski [43].

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