Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 6, NUMBER 4 (2003) ISSN 1311-0454

MELLIN TRANSFORM AND SUBORDINATION LAWS IN FRACTIONAL DIFFUSION PROCESSES

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Dedicated to Paul Butzer, Professor Emeritus,
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on the occasion of his 75-th birthday (April 15, 2003)

Abstract

The Mellin transform is usually applied in probability theory to the product of independent random variables. In recent times the machinery of the Mellin transform has been adopted to describe the Lévy stable distributions, and more generally the probability distributions governed by generalized diffusion equations of fractional order in space and/or in time. In these cases the related stochastic processes are self-similar and are simply referred to as fractional diffusion processes. In this note, by using the convolution properties of the Mellin transform, we provide some (interesting) integral formulas involving the distributions of these processes that can be interpreted in terms of *subordination laws*.

2000 Mathematics Subject Classification: 26A33, 33C60, 42A38, 44A15, 44A35, 60G18, 60G52

Key Words and Phrases: Random variables, Mellin transform, Mellin-Barnes integrals, stable distributions, subordination, self-similar processes

1. Introduction

The role of the Mellin transform in probability theory is mainly related to the product of independent random variables: in fact it is well-known that the probability density of the product of two independent random variables is given by the Mellin convolution of the two corresponding densities. Less known is their role with respect to the class of the Lévy stable distributions, that was formerly outlined by Zolotarev [38] and Schneider [34], see also [36]. A general class of probability distributions (evolving in time), that includes the Lévy strictly stable distributions, is obtained by solving, through the machinery of the Mellin transform, generalized diffusion equations of fractional order in space and/or in time, see [18, 26]. In these cases the related stochastic processes turn out as self-similar and are referred to as fractional diffusion processes.

In this note, after the essential notions and notations concerning the Mellin transform, we first show the role of the Mellin convolution between probability densities to establish subordination laws related to self-similar stochastic processes. Then, for the fractional diffusion processes we establish a sort of Mellin convolutions between the related probability densities, that can be interpreted as *subordination laws*. This is carried out starting for the representations through Mellin-Barnes integrals of the probability densities.

We point out that our results, being based on simple manipulations, can be understood by non-specialists of transform methods and special functions; however they could be derived through a more general analysis involving the class of higher transcendental functions of Fox H type to which the probability densities arising as fundamental solutions of the fractional diffusion equation belong.

2. The Mellin transform

The Mellin transform of a sufficiently well-behaved function $f(x)$ with $x \in \mathbb{R}^+$ is defined by

$$\mathcal{M}\left\{f(x);s\right\} = f^*(s) = \int_0^{+\infty} f(x)\, x^{s-1}\, dx\,, \quad s \in \mathbb{C}\,, \tag{2.1}$$

when the integral converges. Here we assume $f(x) \in L_{loc}(\mathbb{R}^+)$ according to the most usual approach suitable for applied scientists. The basic properties of the Mellin transform follow immediately from those of the bilateral Laplace transform since the two transforms are intimately connected.

Recently the theory of the Mellin transform has been the object of intensive researches by Professor Butzer and his associates, see *e.g.* [6, 7, 8, 9, 10, 11]; in particular Butzer and Jansche [6, 7] have introduced a theory independent from Laplace or Fourier transforms.

The integral (2.1) defines the Mellin transform in a vertical strip in the s-plane whose boundaries are determined by the analytic structure of $f(x)$ as $x \to 0^+$ and $x \to +\infty$.

If we suppose that

$$f(x) = \begin{cases} O\left(x^{-\gamma_1-\epsilon}\right) & \text{as } \; x \to 0^+\,, \\ \\ O\left(x^{-\gamma_2-\epsilon}\right) & \text{as } \; x \to +\infty\,, \end{cases} \tag{2.2}$$

for every (small) $\epsilon > 0$ and $\gamma_1 < \gamma_2$, the the integral (2.1) converges absolutely and defines an analytic function in the strip $\gamma_1 < \Re s < \gamma_2$. This strip is known as the *strip of analyticity* of $\mathcal{M}\{f(x); s\} = f^*(s)$

The inversion formula for (2.1) follows directly from the corresponding inversion formula for the bilateral Laplace transform. We have

$$\mathcal{M}^{-1}\{f^*(s); x\} = f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s)\, x^{-s}\, ds\,, \quad \gamma_1 < \gamma < \gamma_2\,, \tag{2.3}$$

at all points $x \geq 0$ where $f(x)$ is continuous.

Let us now consider the most relevant *Operational Rules*. Denoting by $\stackrel{\mathcal{M}}{\leftrightarrow}$ the juxtaposition of a function $f(x)$ on $x > 0$ with its Mellin transform $f^*(s)$, we have

$$x^a\, f(x) \stackrel{\mathcal{M}}{\longleftrightarrow} f^*(s+a)\,, \quad a \in \mathbb{C}\,,$$

$$f(x^b) \stackrel{\mathcal{M}}{\longleftrightarrow} \frac{1}{|b|} f^*(s/b)\,, \quad b \in \mathbb{C}\,, \quad b \neq 0\,,$$

$$f(cx) \stackrel{\mathcal{M}}{\longleftrightarrow} c^{-s}\, f^*(s)\,, \quad c \in \mathbb{R}\,, \quad c > 0\,,$$

from which

$$x^a\, f(cx^b) \stackrel{\mathcal{M}}{\longleftrightarrow} \frac{1}{|b|}\, c^{-(s+a)/b}\, f^*\left(\frac{s+a}{b}\right)\,. \tag{2.4}$$

Furthermore we have

$$h(x) = \int_0^\infty f\left(\frac{x}{\xi}\right)\, g(\xi)\, \frac{d\xi}{\xi} \stackrel{\mathcal{M}}{\longleftrightarrow} f^*(s)\, g^*(s) = h^*(s)\,, \tag{2.5}$$

which is known as the *Mellin convolution* formula.

3. Subordination in stochastic processes via Mellin convolution

In recent years a number of papers have appeared where explicitly or implicitly subordinated stochastic processes have been treated in view of their relevance in physical and financial applications, see e.g. [1, 2, 3, 30, 31, 33, 35, 36, 37] and references therein. Historically, the notion of subordination was originated by Bochner, see [4, 5]. In brief, according to Feller [17], a *subordinated process* $X(t) = Y(T(t))$ is obtained by randomizing the time clock of a stochastic process $Y(\tau)$ using a new clock $T(t)$, where $T(t)$ is a random process with non-negative independent increments. The resulting process $Y(T(t))$ is said to be subordinated to $Y(\tau)$, called the *parent process*, and is directed by $T(t)$ called the *directing process*. The directing process is often referred to as the operational time. In particular, assuming $Y(\tau)$ to be a Markov process with a spatial probability density function (*pdf*) of x, evolving in time τ, $q_\tau(x) \equiv q(x;\tau)$, and $T(t)$ to be a process with non-negative independent increments with *pdf* of τ depending on a parameter t, $u_t(\tau) \equiv u(\tau;t)$, then the subordinated process $X(t) = Y(T(t))$ is governed by the spatial *pdf* of x evolving with t, $p_t(x) \equiv p(x;t)$, given by the integral representation

$$p_t(x) = \int_0^\infty q_\tau(x)\, u_t(\tau)\, d\tau\,. \tag{3.1}$$

If the parent process $Y(\tau)$ is *self-similar* of the kind that its *pdf* $q_\tau(x)$ is such that

$$q_\tau(x) \equiv q(x;\tau) = \tau^{-\gamma}\, q\left(\frac{x}{\tau^\gamma}\right)\,, \quad \gamma > 0\,, \tag{3.2}$$

then Eq. (3.1) reads,

$$p_t(x) = \int_0^\infty q_\tau\left(\frac{x}{\tau^\gamma}\right) u_t(\tau) \frac{d\tau}{\tau^\gamma}\,. \tag{3.3}$$

Herewith we show how to interpret Eq. (3.3) in terms of a special convolution integral in the framework of the theory of the Mellin transform. Later, in the next sections, we shall show how to use the tools of the Mellin-Barnes integral and Mellin transform to treat the subordination for the class of self-similar stochastic process, which are governed by fractional diffusion equations.

Let X_1 and X_2 be two real *independent* random variables with *pdf*'s $p_1(x_1)$ and $p_2(x_2)$ respectively, with $x_1 \in I\!R$ and $x_2 \in I\!R_0^+$. The joint probability is

$$p_*(x_1, x_2) = p_1(x_1)\, p_2(x_2)\,. \tag{3.4}$$

Denoting by X the random variable obtained by the product of X_1 and X_2^γ, i.e. $x = x_1\, x_2^\gamma$, and carrying out the transformation

$$\begin{cases} x_1 = x/\tau^\gamma\,, \\ x_2 = \tau\,, \end{cases} \tag{3.5}$$

we get the identity

$$\widetilde{p_*}(x,\tau)\, dx\, d\tau = p_1(x/\tau^\gamma)\, p_2(\tau)\, J\, dx\, d\tau\,, \tag{3.6}$$

where

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial x} & \dfrac{\partial x_1}{\partial x_2} \\ \dfrac{\partial x_2}{\partial x} & \dfrac{\partial x_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{\tau^\gamma} & -\dfrac{\gamma x}{\tau^{\gamma+1}} \\ 0 & 1 \end{vmatrix} \tag{3.7}$$

is the Jacobian of the transformation (3.5). Noting that $J = 1/\tau^\gamma$ and integrating (3.6) in $d\tau$ we finally get the *pdf* of X,

$$p(x) = \int_{I\!R^+} p_1\left(\frac{x}{\tau^\gamma}\right) p_2(\tau)\, \frac{d\tau}{\tau^\gamma}\,, \quad x \in I\!R\,. \tag{3.8}$$

For $\gamma = 1$, by comparing with Eq. (2.5), we recover the well known result that the probability density of the product of two independent random variables is given by the Mellin convolution of the two corresponding densities.

We now adapt Eq. (3.8) to our subordination formula (3.3) by identifying p_1, p_2 with q_τ and u_t, respectively.

We can now interpret the subordination formula (3.3) as follows. The *pdf* of the subordinated process X, $p_t(x)$, turns out to be the *pdf* of the product of the independent random variables X_q and ${X_u}^\gamma$ distributed according to $q_\tau(x_q)$ and $u_t(x_u)$, respectively.

4. Fractional diffusion equation and probability distributions

An interesting way to generalize the classical diffusion equation

$$\frac{\partial^2}{\partial x^2}\, u(x,t) \;=\; \frac{\partial}{\partial t}\, u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \ge 0\,, \tag{4.1}$$

is to replace in (4.1) the partial derivatives in space and time by suitable linear integro-differential operators, to be intended as derivatives of non integer order, that allow the corresponding Green function (see below) to be still interpreted as a *spatial probability density evolving in time with an appropriate similarity law.*

<u>*The Space-Time Fractional Diffusion Equation*</u>

Recalling the approach by Mainardi, Luchko and Pagnini in [26], to which we refer the interested reader for details, it turns out that this generalized diffusion equation, that we call *space-time fractional diffusion equation*, is

$$_xD_\theta^\alpha\, u(x,t) \;=\; {_tD_*^\beta}\, u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \ge 0\,, \tag{4.2}$$

where the α , θ , β are real parameters restricted as follows

$$0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,, \tag{4.3}$$

$$0 < \beta \le 1 \quad \text{or} \quad 1 < \beta \le \alpha \le 2\,. \tag{4.4}$$

Here $_xD_\theta^\alpha$ and $_tD_*^\beta$ are integro-differential operators, the *Riesz-Feller space fractional derivative* of order α and asymmetry θ and the *Caputo time fractional derivative* of order β, respectively.

The relevant cases of the space-time fractional diffusion equation (4.2) include, in addition to the standard case of *normal diffusion* $\{\alpha = 2, \beta = 1\}$, the limiting case of the *D'Alembert wave* equation $\{\alpha = 2, \beta = 2\}$, the *space fractional diffusion* $\{0 < \alpha < 2, \beta = 1\}$, the *time fractional diffusion* $\{\alpha = 2, 0 < \beta < 2\}$, and the *neutral fractional diffusion* $\{0 < \alpha = \beta < 2\}$. When $1 < \beta \le 2$ we speak more properly about the *fractional diffusion-wave equation* in that the corresponding equation governs intermediate phenomena between diffusion and wave processes.

Let us now resume the essential definitions of the fractional derivatives in (4.2)-(4.4) based on their Fourier and Laplace representations.

By denoting the Fourier transform of a sufficiently well-behaved (generalized) function $f(x)$ as $\widehat{f}(\kappa) = \mathcal{F}\{f(x);\kappa\} = \int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x} f(x)\,dx$, $\kappa \in \mathbb{R}$, the *Riesz-Feller* space fractional derivative of order α and skewness θ is defined as

$$\begin{gathered}\mathcal{F}\{\,{}_xD_\theta^\alpha\, f(x);\kappa\} = -\psi_\alpha^\theta(\kappa)\ \widehat{f}(\kappa)\,,\\ \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha\, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,,\quad 0<\alpha\le 2\,,\quad |\theta| \le \min\{\alpha, 2-\alpha\}\,.\end{gathered} \tag{4.5}$$

In other words the symbol of the pseudo-differential operator ${}_xD_\theta^\alpha$ is the logarithm of the characteristic function of the generic strictly stable probability density according to the Feller parameterization [16, 17], as revisited by Gorenflo and Mainardi [21]. For this density we write

$$L_\alpha^\theta(x) \stackrel{\mathcal{F}}{\longleftrightarrow} \widehat{L_\alpha^\theta}(\kappa) = \exp\left[-\Psi_\alpha^\theta(\kappa)\right]\,, \tag{4.6}$$

where α is just the *stability exponent* ($0<\alpha\le 2$) and θ is a real parameter related to the asymmetry ($|\theta| \le \min\{\alpha, 2-\alpha\}$) improperly called *skewness*.

By denoting the Laplace transform of a sufficiently well-behaved (generalized) function $f(t)$ as $\widetilde{f}(s) = \mathcal{L}\{f(t);s\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt$, $\Re(s) > a_f$, the *Caputo* time fractional derivative of order β ($m-1<\beta\le m$, $m \in \mathbb{N}$) is defined through(1)

$$\mathcal{L}\left\{{}_tD_*^\beta\, f(t);s\right\} = s^\beta\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k}\, f^{(k)}(0^+)\,,\quad m-1<\beta\le m\,. \tag{4.7}$$

This leads to define, see *e.g.* [20, 32],

$${}_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\beta+1-m}}, & m-1<\beta<m,\\ \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,. \end{cases} \tag{4.8}$$

(1) The reader should observe that the *Caputo* fractional derivative introduced in [12, 13, 14] represents a sort of regularization in the time origin for the classical *Riemann-Liouville* fractional derivative see *e.g.* [20, 32]. We note that the *Caputo* fractional derivative coincides with that introduced (independently of Caputo) by *Djrbashian & Nersesian* [15], which has been adopted by Kochubei, see *e.g.* [23, 24] for treating initial value problems in the presence of fractional derivatives. In [11] the authors have pointed out that such derivative was also considered by Liouville himself, but it should be noted that it was disregarded by Liouville who did not recognize its role.

The Green Function

When the diffusion equations (4.1), (4.2) are equipped by the initial and boundary conditions

$$u(x, 0^+) = \varphi(x)\,, \quad u(\pm\infty, t) = 0\,, \tag{4.9}$$

their solution reads $u(x,t) = \int_{-\infty}^{+\infty} G(\xi, t)\, \varphi(x-\xi)\, d\xi$, where $G(x,t)$ denotes the fundamental solution (known as the *Green function*) corresponding to $\varphi(x) = \delta(x)$, the Dirac generalized function. We note that when $1 < \beta \le 2$ we must add a second initial condition $u_t(x, 0^+) = \psi(x)$, which implies two Green functions corresponding to $\{u(x,0^+) = \delta(x),\, u_t(x,0^+) = 0\}$ and $\{u(x,0^+) = 0,\, u_t(x,0^+) = \delta(x)\}$. Here we restrict ourselves to consider the first Green function because only for this it is legitimate to demand it to be a spatial probability density evolving in time, see below.

It is straightforward to derive from (4.2) the composite Fourier-Laplace transform of the Green function by taking into account the Fourier transform for the *Riesz-Feller* space fractional derivative, see (4.5),, and the Laplace transform for the *Caputo* time fractional derivative, see (4.7). We have, see [26]

$$\widehat{\widetilde{G^\theta_{\alpha,\beta}}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \psi^\theta_\alpha(\kappa)}\,. \tag{4.10}$$

By using the known scaling rules for the Fourier and Laplace transforms, we infer without inverting the two transforms,

$$G^\theta_{\alpha,\beta}(x,t) = t^{-\gamma}\, K^\theta_{\alpha,\beta}(x/t^\gamma)\,, \quad \gamma = \beta/\alpha\,, \tag{4.11}$$

where the one-variable function $K^\theta_{\alpha,\beta}$ is the *reduced Green function* and x/t^γ is the *similarity variable*. We note from $\widehat{\widetilde{G^\theta_{\alpha,\beta}}}(0,s) = 1/s \iff \widehat{G^\theta_{\alpha,\beta}}(0,t) = 1$, the *normalization property* $\int_{-\infty}^{+\infty} G^\theta_{\alpha,\beta}(x,t)\, dx = \int_{-\infty}^{+\infty} K^\theta_{\alpha,\beta}(x)\, dx = 1$, and, from $\psi^\theta_\alpha(\kappa) = \overline{\psi^\theta_\alpha(-\kappa)} = \psi^{-\theta}_\alpha(-\kappa)$, the *symmetry relation* $K^\theta_{\alpha,\beta}(-x) = K^{-\theta}_{\alpha,\beta}(x)$, allowing us to restrict our attention to $x \ge 0$.

For $1 < \beta \le 2$ we can show, see *e.g.* [28], that the second Green function is a primitive (with respect to the variable t) of the first Green function (4.11), so that, being no longer normalized in $\mathbb{R}$, cannot be interpreted as a spatial probability density.

When $\alpha = 2$ ($\theta = 0$) and $\beta = 1$ the inversion of the Fourier-Laplace transform in (4.10) is trivial: we recover the Gaussian density, evolving in time with variance $\sigma^2 = 2t$, typical of the normal diffusion,

$$G^0_{2,1}(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-x^2/(4t)\right)\,, \quad x \in \mathbb{R}\,, \quad t > 0\,, \tag{4.12}$$

which exhibits the similarity law (4.11) with $\gamma = 1/2$.

For the analytical and computational determination of the reduced Green function we restrict our attention to $x > 0$ because of the *symmetry relation.* In this range Mainardi, Luchko and Pagnini [26] have provided the Mellin-Barnes[(2)] integral representation

$$K^{\theta}_{\alpha,\beta}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})\,\Gamma(1-s)}{\Gamma(1-\frac{\beta}{\alpha}s)\,\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\, x^{\,s}\, ds\,,\ \rho = \frac{\alpha-\theta}{2\,\alpha}\,, \quad (4.13)$$

where γ is a suitable real constant.

<u>*The Space Fractional Diffusion*</u> : $\{0 < \alpha < 2\,,\, \beta = 1\}$

In this case we recover the class $L^{\theta}_{\alpha}(x)$ of the strictly stable (non-Gaussian) densities exhibiting fat tails (with the algebraic decay proportional to $|x|^{-(\alpha+1)}$) and infinite variance,

$$K^{\theta}_{\alpha,1}(x) = L^{\theta}_{\alpha}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha)\,\Gamma(1-s)}{\Gamma(\rho\, s)\,\Gamma(1-\rho s)}\, x^{\,s}\, ds\,,\ \rho = \frac{\alpha-\theta}{2\,\alpha}\,, \quad (4.14)$$

where $0 < \gamma < \min\{\alpha, 1\}$.

A stable *pdf* with extremal value of the skewness parameter is called *extremal.* One can prove that all the extremal stable *pdf*s' with $0 < \alpha < 1$ are one-sided, the support being $\mathbb{R}^+_0$ if $\theta = -\alpha$, and $\mathbb{R}^-_0$ if $\theta = +\alpha$. The one-sided stable *pdf*'s with support in $\mathbb{R}^+_0$ can be better characterized by their (spatial) Laplace transform, which turn out to be

$$\widetilde{L^{-\alpha}_{\alpha}}(s) := \int_0^{\infty} \mathrm{e}^{\,-sx}\, L^{-\alpha}_{\alpha}(x)\, dx = \mathrm{e}^{\,-s^{\alpha}}\,, \quad \Re\,(s) > 0\,, \quad 0 < \alpha < 1\,. \qquad (4.15)$$

In terms of Mellin-Barnes integral representation we have

$$L^{-\alpha}_{\alpha}(x) = \frac{1}{\alpha x}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha)}{\Gamma(s)}\, x^{\,s}\, ds\,, \quad 0 < \gamma < \alpha < 1\,. \qquad (4.16)$$

(2) The names refer to the two authors, who in the first 1910's developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, these integrals were first used by S. Pincherle in 1888. For a revisited analysis of the pioneering work by Pincherle (1853-1936, Professor of Mathematics at the University of Bologna from 1880 to 1928) we refer to Mainardi and Pagnini [27].

<u>*The Time Fractional Diffusion*</u> : $\{\alpha = 2\,,\, 0 < \beta < 2\}$

In this case, we recover the class of the Wright type densities exhibiting stretched exponential tails and finite variance proportional to t^β ,

$$K^0_{2,\beta}(x) = \frac{1}{2}M_{\beta/2}(x) = \frac{1}{2x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-\beta s/2)}\, x^{\,s}\, ds\,, \tag{4.17}$$

where $0 < \gamma < 1$. As a matter of fact, $\frac{1}{2}M_{\beta/2}(x)$ turns out to be a symmetric probability density related to the transcendental function $M_\nu(z)$ defined for any $\nu \in (0,1)$ and $\forall z \in \mathbb{C}$ as

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (1-\nu)]} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\Gamma(\nu n)\sin(\pi\nu n). \tag{4.18}$$

We note that $M_\nu(z)$ is an entire function of order $\rho = 1/(1-\nu)$, that turns out to be a special case of the Wright function(3). Restricting our attention to the positive real axis ($r \geq 0$) we have: the *Laplace transform*

$$\mathcal{L}\{M_\nu(r); s\} = E_\nu(-s)\,, \quad \Re\,(s) > 0\,, \quad 0 < \nu < 1\,, \tag{4.19}$$

where E_ν is the Mittag-Leffler function, and the *asymptotic representation*

$$\begin{gathered} M_\nu(r) \sim A_0\, Y^{\,\nu-1/2} \exp\,(-Y)\,, \quad x \to \infty\,, \\ A_0 = \frac{1}{\sqrt{2\pi}\,(1-\nu)^\nu\, \nu^{2\nu-1}}\,, \quad Y = (1-\nu)\,(\nu^\nu\, r)^{1/(1-\nu)}\,, \end{gathered} \tag{4.20}$$

a result formerly obtained by Wright himself, and, independently, by Mainardi and Tomirotti [29] by using the saddle point method. Because of the above exponential decay, any moment of order $\delta > -1$ for $M_\nu(r)$ is finite and given as

$$\int_0^\infty r^{\,\delta}\, M_\nu(r)\, dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}\,, \quad \delta > -1\,. \tag{4.21}$$

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(3) The Wright function is defined by the series representation, valid in the whole complex plane,

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n + \mu)}\,, \quad \lambda > -1\,, \quad \mu \in \mathbb{C}\,, \quad z \in \mathbb{C}\,.$$

Then, $M_\nu(z) := \Phi_{-\nu,1-\nu}(-z)$ with $0 < \nu < 1$. The function $M_\nu(z)$ provides a generalization of the Gaussian and of the Airy function in that

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-\,z^2/4\right)\,, \quad M_{1/3}(z) = 3^{2/3}\,\mathrm{Ai}\left(z/3^{1/3}\right)\,.$$

In particular we get the normalization property in $I\!R^+$, $\int_0^\infty M_\nu(r)\,dr = 1$. In view of Eqs (4.11) and (4.21)the moments (of even order) of the fundamental solution $G^0_{2,\beta}(x,t)$ turn out to be, for $n = 0, 1, 2, \ldots$ and $t \ge 0$,

$$\int_{-\infty}^{+\infty} x^{2n}\, G^0_{2,\beta}(x,t)\,dx = t^{\beta n} \int_0^\infty x^{2n} M_{\beta/2}(x)\,dx = \frac{\Gamma(2n+1)}{\Gamma(\beta n+1)}\, t^{\beta n}. \tag{4.22}$$

We agree to call $M_\nu(r)$ ($\nu \in (0,1)$, $r \in I\!R_0^+$) the *M-Wright function of order* ν, understanding that its half represents the spatial *pdf* corresponding to the *time fractional diffusion* equation of order 2ν. Relevant properties of this function, see *e.g.* [19, 25, 26], are concerning the limit expression for $\beta = 1$, i.e. $M_1(r) = \delta(r-1)$ and its relation with the extremal stable densities, i.e.

$$\frac{1}{c^{1/\nu}}\, L_\nu^{-\nu}\left(\frac{r}{c^{1/\nu}}\right) = \frac{c\,\nu}{r^{\nu+1}}\, M_\nu\left(\frac{c}{r^\nu}\right), \quad 0 < \nu < 1\,, \quad c > 0\,, \quad r > 0\,. \tag{4.23}$$

We note that, in both limiting cases of space fractional ($\alpha = 2$) and time fractional ($\beta = 1$) diffusion, we recover the Gaussian density of the *normal diffusion*, for which

$$\begin{aligned} K^0_{2,1}(x) &= \frac{1}{2x}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-s/2)}\, x^{\,s}\,ds \quad (0<\gamma<1) \\ &= L^0_2(x) = \frac{1}{2} M_{1/2}(x) = \frac{1}{2\sqrt{\pi}}\,\mathrm{e}^{-x^2/4}\,. \end{aligned} \tag{4.24}$$

<u>*The Neutral Fractional Diffusion*</u> : $\{0 < \alpha = \beta < 2\}$

In this case, surprisingly, the corresponding (reduced) Green function can be expressed (in explicit form) in terms of a (non-negative) simple elementary function, that we denote by $N^\theta_\alpha(x)$, as it is shown in [26]:

$$\begin{aligned} N^\theta_\alpha(x) &= \frac{1}{\alpha x}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})}{\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\, x^{\,s}\,ds \\ &= \frac{1}{\pi}\,\frac{x^{\alpha-1}\sin[\frac{\pi}{2}(\alpha-\theta)]}{1+2x^\alpha\cos[\frac{\pi}{2}(\alpha-\theta)]+x^{2\alpha}}\,. \end{aligned} \tag{4.25}$$

<u>*The Fractional Diffusion Processes*</u>

The *self-similar stochastic processes* generated by the above probability densities evolving in time can be considered as generalizations of the standard diffusion processes and therefore distinguished from it with the label "fractional". When $0 < \beta < 1$ random walk models can be introduced to generalize the classical

Brownian motion of the standard diffusion, as it was investigated in a number of papers of our group see *e.g.* [21, 22]. In the case of *space fractional diffusion* we obtain a special class of Markovian processes, called stable Lévy motions, which exhibit infinite variance associated to the possibility of arbitrarily large jumps (*Lévy flights*). In the case of *time fractional diffusion*, we obtain a class of stochastic processes which are non-Markovian and exhibit a variance consistent with slow anomalous diffusion. For the general genuine *space-time fractional* diffusion ($0 < \alpha < 2\,,\ 0 < \beta < 1$), we generate a class of densities (symmetric or non-symmetric according to $\theta = 0$ or $\theta \neq 0$) which exhibit fat tails with an algebraic decay $\propto |x|^{-(\alpha+1)}$. Thus they belong to the domain of attraction of the Lévy stable densities of index α and can be referred to as *fractional stable densities.* The related stochastic processes possess the characteristics of the previous two classes; indeed, they are non-Markovian (being $0 < \beta < 1$) and exhibit infinite variance associated to the possibility of arbitrarily large jumps (being $0 < \alpha < 2$).

5. Subordination for space fractional diffusion processes

In the book by Feller, see [17], at p. 176 we read: *Let X and Y be independent strictly stable variables, with characteristic exponent α and β respectively. Assume Y to be a positive variable (whence $\beta < 1$). The product $X\,Y^{1/\alpha}$ has a stable distribution with exponent $\alpha\,\beta$.*

In other words, this statement means that any strictly stable process (of exponent $\gamma = \alpha \cdot \beta$) is subordinated to a parent strictly stable process (of exponent α) and directed by an extremal strictly stable process (of exponent $0 < \beta < 1$). Feller's proof is vague being, as a matter of fact, limited to *symmetric* subordinated and parent stable distributions. Furthermore, the proof, scattered in several sections, is essentially based on the use of Fourier and Laplace transforms. Here we would like to make more precise the previous statement by Feller by considering the possibility of asymmetry characterized by the index θ as previously explained and making use of the Mellin machinery outlined in Sections 2 and 3, and of the Mellin-Barnes integral representation (4.14). So, in virtue of the Mellin inversion formula (2.3) we can write for the generic *strictly stable pdf*

$$L_\alpha^\theta(x) \stackrel{\mathcal{M}}{\longleftrightarrow} \frac{1}{\alpha}\frac{\Gamma\left(-\frac{s-1}{\alpha}\right)\Gamma[1+(s-1)]}{\Gamma[1+\rho(s-1)]\Gamma[-\rho(s-1)]}\,, \quad \rho = \frac{\alpha-\theta}{2\,\alpha}\,. \tag{5.1}$$

Let us now consider the evolution in time according to the *space fractional diffusion* equation by writing

$$L_\alpha^\theta(x;t) := G_{\alpha,1}^\theta(x,t) = t^{-1/\alpha}\, L_\alpha^\theta\left(\frac{x}{t^{1/\alpha}}\right)\,. \tag{5.2}$$

Then we prove the following

THEOREM. *Let $L_{\alpha_p}^{\theta_p}(x;t)$, $L_{\alpha_q}^{\theta_q}(x;t)$ and $L_{\beta}^{\theta_\beta}(x;t)$ be strictly stable densities with exponents $\alpha_p\,, \alpha_q\,, \beta$ and asymmetry parameters $\theta_p\,, \theta_q\,, \theta_\beta$, respectively, such that*

$$0 < \alpha_p \le 2\,, \quad |\theta_p| \le \min\{\alpha_p, 2-\alpha_p\}\,,$$
$$0 < \alpha_q \le 2\,, \quad |\theta_q| \le \min\{\alpha_q, 2-\alpha_q\}\,,$$
$$0 < \beta \le 1\,, \quad \theta_\beta = -\beta\,,$$

then the following subordination formula holds true for $0 < x < \infty$,

$$L_{\alpha_p}^{\theta_p}(x;t) = \int_0^\infty L_{\alpha_q}^{\theta_q}(x;\tau)\, L_{\beta}^{-\beta}(\tau;t)\, d\tau\,, \text{ with } \alpha_p = \beta\alpha_q\,,\ \theta_p = \beta\theta_q\,. \tag{5.3}$$

Because of the scaling property (5.2) of the stable pdf's, we can alternatively state

$$t^{-1/\alpha_p} L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right) = \int_0^\infty \tau^{-1/\alpha_q} L_{\alpha_q}^{\theta_q}\left(\frac{x}{\tau^{1/\alpha_q}}\right) t^{-1/\beta} L_{\beta}^{-\beta}\left(\frac{\tau}{t^{1/\beta}}\right) d\tau\,. \tag{5.4}$$

The proof of Eq. (5.4) is a (straightforward) consequence of the previous considerations. By recalling the Mellin pairs for the involved stable densities (that can be easily obtained from Eq. (5.1)-(5.2) by adopting the correct parameters) and the scaling properties of the Mellin transform, see (2.4), we have

$$t^{-1/\alpha_p} L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right) \stackrel{\mathcal{M}}{\longleftrightarrow} t^{-1/\alpha_p}\left(\frac{1}{t^{1/\alpha_p}}\right)^{-s} \frac{1}{\alpha_p} \frac{\Gamma\left(-\frac{s-1}{\alpha_p}\right)\Gamma[1+(s-1)]}{\Gamma[1+\rho_p(s-1)]\Gamma[-\rho_p(s-1)]} \tag{5.5}$$

and

$$b\,c\,x^a\, L_{\beta}^{\theta_\beta}(cx^b) \stackrel{\mathcal{M}}{\longleftrightarrow} c^{1-\frac{s+a}{b}} \frac{1}{\beta} \frac{\Gamma\left(-\frac{s+a}{b\beta}+\frac{1}{\beta}\right)\Gamma\left[1+\left(\frac{s+a}{b}-1\right)\right]}{\Gamma\left[1+\rho_\beta\left(\frac{s+a}{b}-1\right)\right]\Gamma\left[-\rho_\beta\left(\frac{s+a}{b}-1\right)\right]}\,. \tag{5.6}$$

After some algebra we recognize

$$\mathcal{M}\{t^{-1/\alpha_p}\, L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right); s\} = \mathcal{M}\{bx^a\, c\, L_{\beta}^{\theta_\beta}(cx^b); s\}\, \mathcal{M}\{L_{\alpha_q}^{\theta_q}(x); s\} \tag{5.7}$$

provided that

$$\theta_\beta = -\beta\,, \quad a = \alpha_q - 1\,, \quad b = \alpha_q\,, \quad c = t^{-1/\beta}\,, \tag{5.8}$$

and

$$\alpha_p = \beta\alpha_q\,, \quad \theta_p = \beta\theta_q\,. \tag{5.9}$$

Recalling the Mellin convolution formula (2.5) we obtain from Eqs. (5.7)-(5.9) the integral representation

$$t^{-1/\alpha_p}\, L_{\alpha_p}^{\theta_p}\left(\frac{x}{t^{1/\alpha_p}}\right) = \int_0^\infty \alpha_q \xi^{\alpha_q-1}\, L_{\alpha_q}^{\theta_q}\left(\frac{x}{\xi}\right) t^{-1/\beta}\, L_{\beta}^{-\beta}\left(\frac{\xi^{\alpha_q}}{t^{1/\beta}}\right) \frac{d\xi}{\xi}\,, \tag{5.10}$$

and, by replacing $\xi \to \tau^{1/\alpha_q}$, we finally get Eq. (5.4).

Taking into account the relationships in Eq. (5.9), we can point out some interesting *subordination laws.* In particular we observe that any *symmetric* stable distribution with exponent $\alpha_p = \alpha \in (0,2)$ ($\theta_p = \theta = 0$) is subordinated to the *Gaussian distribution* ($\alpha_q = 2\,,\, \theta_q = 0$), see $L_2^0(x)$ in (4.24), through an extremal stable density of exponent $\beta = \alpha/2$, that is

$$L_\alpha^0(x;t) = \int_0^\infty L_2^0(x;\tau)\, L_{\alpha/2}^{-\alpha/2}(\tau;t)\, d\tau\,, \quad 0 < \alpha < 2\,. \tag{5.11}$$

Furthermore, by recalling the *generalized Cauchy density* of skewness θ ($|\theta| < 1$), see *e.g.* Eq. (4.9) in [26],

$$L_1^\theta(x) = \frac{1}{\pi} \frac{\cos(\theta\pi/2)}{[x + \sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2}\,, \ |\theta| < 1\,,\ -\infty < x < +\infty\,. \tag{5.12}$$

we note that any stable distribution with exponent $\alpha_p = \alpha \in (0,1)$ and skewness $|\theta_p| = |\theta| < \alpha$ is subordinated to the *generalized Cauchy distribution* with skewness $|\theta_q| = |\theta|/\alpha < 1$ denoted by $L_1^{\theta/\alpha}(x)$, see (5.12), through an extremal density of exponent $\beta = \alpha$, that is

$$L_\alpha^\theta(x;t) = \int_0^\infty L_1^{\theta/\alpha}(x;\tau)\, L_\alpha^{-\alpha}(\tau;t)\, d\tau\,. \tag{5.13}$$

6. Subordination for time fractional diffusion processes

For the *M-Wright function* we deduce from (4.17) the Mellin transform pair:

$$M_\nu(r) \overset{\mathcal{M}}{\longleftrightarrow} \frac{\Gamma\,[1 + (s-1)]}{\Gamma[1 + \nu(s-1)]}\,, \quad 0 < \nu < 1\,. \tag{6.1}$$

Let us now consider the evolution in time according to the corresponding *time fractional diffusion* equation by writing

$$M_\nu(x;t) := 2\, G_{2,2\nu}^0(x,t) = t^{-\nu}\, M_\nu\left(\frac{x}{t^\nu}\right)\,. \tag{6.2}$$

Then we prove the following

THEOREM. *Let $M_\nu(x;t)$, $M_\eta(x;t)$ and $M_\beta(x;t)$ be M-Wright functions of orders $\nu, \eta, \beta \in (0,1)$ respectively, then the following subordination formula holds true for $0 < x < \infty$,*

$$M_\nu(x,t) = \int_0^\infty M_\eta(x;\tau)\, M_\beta(\tau;t)\, d\tau\,, \ \textit{with} \ \ \nu = \eta\,\beta\,. \tag{6.3}$$

Because of the scaling property (6.2) of the M-Wright functions, we can alternatively state

$$t^{-\nu}\, M_\nu\left(\frac{x}{t^\nu}\right) = \int_0^\infty \tau^{-\eta}\, M_\eta\left(\frac{x}{\tau^\eta}\right) t^{-\beta}\, M_\beta\left(\frac{\tau}{t^\beta}\right) d\tau\,. \tag{6.4}$$

The proof of Eq. (6.4) is a (straightforward) consequence of the previous considerations. After some algebra we recognize

$$\mathcal{M}\{t^{-\nu}\, M_\nu\left(\frac{x}{t^\nu}\right); s\} = \mathcal{M}\{b\, c\, x^a\, M_\beta(c\, x^b); s\}\, \mathcal{M}\{M_\eta(x); s\} \tag{6.5}$$

provided that

$$a = \frac{1}{\eta} - 1\,, \quad b = \frac{1}{\eta}\,, \quad c = t^{-\nu/\eta}\,, \tag{6.6}$$

and

$$\nu = \beta\, \eta\,. \tag{6.7}$$

Recalling the Mellin convolution formula (2.5) we obtain from Eqs. (6.5)-(6.7) the integral representation

$$t^{-\nu}\, M_\nu\left(\frac{x}{t^\nu}\right) = \int_0^\infty \frac{1}{\eta}\, \xi^{1/\eta-1}\, M_\eta\left(\frac{x}{\xi}\right) t^{-\beta}\, M_\beta\left(\frac{\xi^{1/\eta}}{t^\beta}\right) \frac{d\xi}{\xi}\,, \tag{6.8}$$

and, by replacing $\xi \to \tau^\eta$, we finally get Eq. (6.4).

We note that for $\eta = 1/2$ the corresponding M-Wright function reduces to twice the Gaussian density according to (4.24), so the subordination formula (6.3) reads

$$M_{\beta/2}(x; t) = 2\int_0^\infty L_2^0(x; \tau)\, M_\beta(\tau; t)\, d\tau\,. \tag{6.9}$$

7. Subordination for space-time fractional diffusion

For the *space-time fractional* diffusion we shall prove two relevant *subordination laws.* For this purpose we first consider the evolution in time writing

$$K^\theta_{\alpha,\beta}(x; t) := G^\theta_{\alpha,\beta}(x, t) = t^{-\beta/\alpha}\, K^\theta_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right)\,. \tag{7.1}$$

Then the required subordination laws read

$$K^\theta_{\alpha,\beta}(x; t) = \int_0^\infty L^\theta_\alpha(x; \tau)\, M_\beta(\tau; t)\, d\tau\,, \quad 0 < \beta \le 1\,. \tag{7.2}$$

and

$$K^\theta_{\alpha,\beta}(x; t) = \int_0^\infty N^\theta_\alpha(x; \tau)\, M_{\beta/\alpha}(\tau; t)\, d\tau\,, \quad 0 < \beta/\alpha \le 1\,. \tag{7.3}$$

In order to prove the above laws we start from two relevant results in [26], see (6.1), that we report below

$$K_{\alpha,\beta}^{\theta}(x) = \begin{cases} \alpha \displaystyle\int_0^\infty \left[\xi^{\alpha-1}\, M_\beta\,(\xi^\alpha)\right]\, L_\alpha^\theta\,(x/\xi)\,\frac{d\xi}{\xi}\,, & 0 < \beta \le 1\,, \\ \displaystyle\int_0^\infty M_{\beta/\alpha}(\xi)\, N_\alpha^\theta(x/\xi)\,\frac{d\xi}{\xi}\,, & 0 < \beta/\alpha \le 1\,. \end{cases} \tag{7.4}$$

In [26] the above identities have enabled us to *extend the probability interpretation of the Green functions to the ranges* $\{0 < \alpha < 2\} \cap \{0 < \beta < 1\}$ *and* $\{1 < \beta < \alpha < 2\}$. Indeed, the formulae in (7.4) show the non-negativity of the Green functions of the genuine space-time fractional diffusion equation based on the non-negativity of the Green functions of the particular cases of space, time and neutral fractional diffusion. We note that the formulae were derived by using the Mellin-Barnes representation of the corresponding Green functions, a method akin with that of the Mellin transform as we have previously seen.

Let us now prove Eq. (7.2). Because of the scaling properties of the involved functions this is equivalent to prove

$$t^{-\beta/\alpha}\, K_{\alpha,\beta}^\theta\left(\frac{x}{t^{\beta/\alpha}}\right) = \int_0^\infty \tau^{-1/\alpha}\, L_\alpha^\theta\left(\frac{x}{\tau^{1/\alpha}}\right)\, t^{-\beta}\, M_\beta\left(\frac{\tau}{t^\beta}\right)\, d\tau\,,\ 0 < \beta \le 1\,, \tag{7.5}$$

that can be easily achieved from the first equation of (7.4), by replacing x with $x/t^{\beta/\alpha}$ and making the change of variable $\xi = \tau^{1/\alpha}/t^{\beta/\alpha}$ in the integral.

Similarly, to prove Eq. (7.3) we can verify

$$t^{-\beta/\alpha}\, K_{\alpha,\beta}^\theta\left(\frac{x}{t^{\beta/\alpha}}\right) = \int_0^\infty \tau^{-1}\, N_\alpha^\theta\left(\frac{x}{\tau}\right)\, t^{-\beta/\alpha}\, M_{\beta/\alpha}\left(\frac{\tau}{t^{\beta/\alpha}}\right)\, d\tau,\ 0 < \frac{\beta}{\alpha} \le 1. \tag{7.6}$$

In this case it suffices to replace x with $x/t^{\beta/\alpha}$ and set $\xi = \tau/t^{\beta/\alpha}$ in the second equation of (7.4).

It is worth to note that whereas for the particular cases of space and time fractional diffusion the corresponding subordination formulas (5.3) and (6.3) involve functions of the same class, here the subordination formulas (7.2) and (7.3) involve functions of different classes. However, in all the cases we point out that *the pdf of the directing process is a M-Wright function*, including (5.3) if we take into account (4.23).

If we like to use the terminology $G_{\alpha,\beta}^\theta(x;t)$ for the Green functions, the two subordination laws (7.2)-(7.3) can be resumed as follows

$$G_{\alpha,\beta}^\theta(x;t) = \begin{cases} 2\displaystyle\int_0^\infty G_{\alpha,1}^\theta(x;\tau)\, G_{2,2\beta}^0(\tau;t)\, d\tau\,, & 0 < \beta \le 1\,, \\ 2\displaystyle\int_0^\infty G_{\alpha,\alpha}^\theta(x;\tau)\, G_{2,2\beta/\alpha}^0(\tau;t)\, d\tau\,, & 0 < \beta/\alpha \le 1\,, \end{cases} \tag{7.7}$$

From the first equation in (7.7) we note that the Green function for the space-time fractional diffusion equation of order $\{\alpha, \beta\}$, with $0 < \alpha \le 2$ and $0 < \beta \le 1$, can be expressed in terms of the Green function for the space fractional diffusion equation of order α and the Green function for the time fractional diffusion equation of order 2β.

8. Concluding discussion

There are several ways of deriving subordination formulas for fractional diffusion processes. A natural way is to start from an approximating continuous time random walk model and carry out an appropriate passage to the limit. As we cannot give here a comprehensive survey of achievements of contributors to this subject, let us only cite [1, 30] for a concise description. In contrast to this "stochastic" way it was our aim to show how classes of subordination formulas can be found by purely analytic methods, namely by exploiting solution formulas for the fundamental solutions of the essential types of fractional diffusion equations, formulas expressing these solutions in form of Mellin-Barnes integrals (see [26]). Using the machinery offered by Mellin transform theory these formulas can be rewritten as integrals of products that in turn allow a probabilistic interpretation as subordination formulas. So, we testify for the fact that the fascinating field of fractional diffusion processes is not only interesting from the view-points of probability theory and statistical physics, but also from that of pure analysis where it is a playground for lovers of special functions and integral transforms. In particular, we also demonstrate that the Mellin transform is an important and useful integral transform in its own right.

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Received: December 18, 2003

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