SPACE-TIME FRACTIONAL DIFFUSION: EXACT SOLUTIONS AND PROBABILITY INTERPRETATION

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The fundamental solutions (Green functions) for the Cauchy problems of the space-time fractional diffusion equation are investigated with respect to their scaling and similarity properties, starting from their composite Fourier-Laplace representation. By using the Mellin transform, a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane is presented, that allows us to obtain their computational form in the space-time domain and to analyse their probability interpretation.

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1 Introduction

By replacing in the standard diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}\,u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{1.1}$$

where $u = u(x,t)$ is the (real) field variable, the second-order space derivative and the first-order time derivative by suitable *integro-differential* operators, which can be interpreted as a space and time derivative of fractional order, we obtain a sort of "generalized diffusion" equation. Such equation may be referred to as the *space-time fractional diffusion* equation when its fundamental solution (see below) can be interpreted as a probability density. We write

$${}_tD_*^{\beta}\,u(x,t) \;=\; {}_xD_{\theta}^{\alpha}\,u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{1.2}$$

where the $\alpha\,,\,\theta\,,\,\beta$ are real parameters restricted as follows

$$0 < \alpha \leq 2\,, \quad |\theta| \leq \min\{\alpha, 2-\alpha\}\,, \quad 0 < \beta \leq 2\,. \tag{1.3}$$

In (1.2) ${}_xD_\theta^\alpha$ is the space fractional *Riesz-Feller derivative* of order α and skewness θ, and ${}_tD_*^\beta$ is the time fractional *Caputo derivative* of order β. The definitions of these fractional derivatives are more easily understood if given in terms of Fourier transform and Laplace transform, respectively.

For the space fractional *Riesz-Feller derivative* we have

$$\mathcal{F}\{{}_xD_\theta^\alpha f(x);\kappa\} = -\psi_\alpha^\theta(\kappa)\,\widehat{f}(\kappa)\,, \quad \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha\,\mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,, \tag{1.4}$$

where $\kappa \in I\!R$ and $\widehat{f}(\kappa) = \mathcal{F}\{f(x);\kappa\} = \int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x} f(x)\,dx$. In other words the symbol of the pseudo-differential operator[a] ${}_xD_\theta^\alpha$ is required to be the logarithm of the characteristic function of the generic *stable* (in the Lévy sense) probability density, according to the Feller parameterization [3]. For $\alpha = 2$ (hence $\theta = 0$) we have $\widehat{{}_xD_0^2}(\kappa) = -\kappa^2 = (-i\kappa)^2$, so we recover the standard second derivative. More generally for $\theta = 0$ we have $\widehat{{}_xD_0^\alpha}(\kappa) = -|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$ so

$${}_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}. \tag{1.5}$$

In this case we call the LHS of (1.5) simply the *Riesz fractional derivative* operator of order α. For the explicit expressions in integral form of the general *Riesz-Feller fractional derivative* we refer the interested reader to Mainardi, Luchko and Pagnini [9]. Let us now consider the time fractional *Caputo derivative.* Following the original idea by Caputo [1], see also [2,6,12], a proper time fractional derivative of order $\beta \in (m-1, m]$ with $m \in I\!N$, useful for physical applications, may be defined in terms of the following rule for the Laplace transform:

$$\mathcal{L}\left\{{}_tD_*^\beta f(t);s\right\} = s^\beta\,\widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k}\,f^{(k)}(0^+)\,, \quad m-1 < \beta \le m\,, \tag{1.6}$$

where $s \in \mathbb{C}$ and $\widetilde{f}(s) = \mathcal{L}\{f(t);s\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt$. Then the *Caputo fractional derivative* of $f(t)$ turns out to be defined as

$${}_tD_*^\beta f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\beta+1-m}}\,, & m-1 < \beta < m\,, \\ \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,. \end{cases} \tag{1.7}$$

[a] Let us recall that a generic linear pseudo-differential operator A, acting with respect to the variable $x \in I\!R$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x} A[f(x)]\,dx = \widehat{A}(\kappa)\,\widehat{f}(\kappa)$, where $\widehat{A}(\kappa)$ is referred to as symbol of A, given as $\widehat{A}(\kappa) = \left(A\,\mathrm{e}^{-i\kappa x}\right)\,\mathrm{e}^{+i\kappa x}$.

In order to formulate and solve the Cauchy problems for (1.2) we have to select explicit initial conditions concerning $u(x,0^+)$ if $0 < \beta \leq 1$ and $u(x,0^+)$, $u_t(x,0^+)$ if $1 < \beta \leq 2$. If $\phi_1(x)$ and $\phi_2(x)$ denote two given real functions of $x \in I\!R$, the Cauchy problems consist in finding the solution of (1.2) subjected to the additional conditions:

$$u(x,0^+) = \phi_1(x)\,, \quad x \in I\!R\,, \quad \text{if} \quad 0 < \beta \leq 1\,; \tag{1.8a}$$

$$\begin{cases} u(x,0^+) = \phi_1(x)\,, \\ u_t(x,0^+) = \phi_2(x)\,, \end{cases} \quad x \in I\!R\,, \quad \text{if} \quad 1 < \beta \leq 2\,. \tag{1.8b}$$

2 The Green functions

The Cauchy problems can be conveniently treated by making use of the most common integral transforms, *i.e.* the Fourier transform (in space) and the Laplace transform (in time). Indeed, the composite Fourier-Laplace transforms of the solutions of the two Cauchy problems:

(a) $\{(1.2) + (1.8a)\}$ if $0 < \beta \leq 1$, (b) $\{(1.2) + (1.8b)\}$ if $1 < \beta \leq 2$,

turn out to be, by using (1.4) and (1.6) with $m = 1, 2$,

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}\,\widehat{\phi_1}(\kappa)\,, \quad 0 < \beta \leq 1\,, \tag{2.1a}$$

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + \psi_\alpha^\theta(\kappa)}\,\widehat{\phi_1}(\kappa) + \frac{s^{\beta-2}}{s^\beta + \psi_\alpha^\theta(\kappa)}\,\widehat{\phi_2}(\kappa)\,, \quad 1 < \beta \leq 2\,. \tag{2.1b}$$

By fundamental solutions (or Green functions) of the above Cauchy problems we mean the (generalized) solutions corresponding to the initial conditions:

$$G_{\alpha,\beta}^{\theta\,(1)}(x,0^+) = \delta(x)\,, \quad 0 < \beta \leq 1\,; \tag{2.2a}$$

$$\begin{cases} G_{\alpha,\beta}^{\theta\,(1)}(x,0^+) = \delta(x)\,, \\ \dfrac{\partial}{\partial t}\,G_{\alpha,\beta}^{\theta\,(1)}(x,0^+) = 0\,, \end{cases} \quad \begin{cases} G_{\alpha,\beta}^{\theta\,(2)}(x,0^+) = 0\,, \\ \dfrac{\partial}{\partial t}\,G_{\alpha,\beta}^{\theta\,(2)}(x,0^+) = \delta(x)\,, \end{cases} \quad 1 < \beta \leq 2\,. \tag{2.2b}$$

We have denoted by $\delta(x)$ the delta-Dirac generalized function, whose (generalized) Fourier transform is known to be 1, and we have distinguished by the apices (1) and (2) the two types of Green functions. From Eqs (2.1a)-(2.1b) the Fourier-Laplace transforms of these Green functions turn out to be

$$\widehat{\widetilde{G}}{}_{\alpha,\beta}^{\theta\,(j)}(\kappa, s) = \frac{s^{\beta-j}}{s^\beta + \psi_\alpha^\theta(\kappa)}\,, \quad 0 < \beta \leq 2\,, \quad j = 1, 2\,. \tag{2.3}$$

Furthermore, by recalling the Fourier convolution property, we note that the Green functions allow us the represent the solutions of the above two Cauchy problems through the relevant integral formulas:

$$u(x,t) = \int_{-\infty}^{+\infty} G^{\theta\,(1)}_{\alpha,\beta}(\xi,t)\,\phi_1(x-\xi)\,d\xi\,, \quad 0<\beta\le 1\,; \qquad (2.4a)$$

$$u(x,t) = \int_{-\infty}^{+\infty} [G^{\theta\,(1)}_{\alpha,\beta}(\xi,t)\phi_1(x-\xi)+G^{\theta\,(2)}_{\alpha,\beta}(\xi,t)\phi_2(x-\xi)]\,d\xi,\ 1<\beta\le 2. \quad (2.4b)$$

We recognize from (2.3) that the function $G^{\theta\,(2)}_{\alpha,\beta}(x,t)$ along with its Fourier-Laplace transform is well defined also for $0<\beta\le 1$ even if it loses its meaning of being a fundamental solution of (1.2), resulting

$$G^{\theta\,(2)}_{\alpha,\beta}(x,t) = \int_0^t G^{\theta\,(1)}_{\alpha,\beta}(x,\tau)\,d\tau\,, \quad 0<\beta\le 2\,. \qquad (2.5)$$

By using the known scaling rules for the Fourier and Laplace transforms, and introducing the *similarity variable* $x/t^{\beta/\alpha}$, we infer from (2.3) (thus without inverting the two transforms) the *scaling properties* of the Green functions,

$$G^{\theta\,(j)}_{\alpha,\beta}(x,t) = t^{-\beta/\alpha+j-1}\,K^{\theta\,(j)}_{\alpha,\beta}\left(x/t^{\beta/\alpha}\right)\,, \qquad (2.6)$$

where the one-variable functions $K^{\theta\,(1)}_{\alpha,\beta}(x)$, $K^{\theta\,(2)}_{\alpha,\beta}(x)$ are called the *reduced Green functions.* We also note the *symmetry relation*:

$$G^{\theta\,(j)}_{\alpha,\beta}(-x,t) = G^{-\theta\,(j)}_{\alpha,\beta}(x,t)\,, \quad j=1,2\,, \qquad (2.7)$$

so for the determination of the Green functions we can restrict our attention to $x>0$. Extending the method illustrated in [4,9], where only the Green function of type (1) was determined, we first invert the Laplace transforms getting

$$\widehat{G^{\theta\,(j)}_{\alpha,\beta}}(\kappa,t) = t^{j-1}\,E_{\beta,j}[-\psi^\theta_\alpha(\kappa)t^\beta],\ \widehat{K^{\theta\,(j)}_{\alpha,\beta}}(\kappa) = E_{\beta,j}[-\psi^\theta_\alpha(\kappa)],\ j=1,2, \quad (2.8)$$

where $E_{\beta,j}$ denotes the two-parameter Mittag-Leffler function[b]. We note the normalization property $\int_{-\infty}^{+\infty} K^{\theta\,(j)}_{\alpha,\beta}(x)\,dx = E_{\beta,j}(0) = 1/\Gamma(j) = 1$ for $j=1,2$.

[b] The Mittag-Leffler function $E_{\beta,\mu}(z)$ with $\beta,\mu>0$ is an entire transcendental function of order $\rho=1/\beta$, defined in the complex plane by the power series

$$E_{\beta,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+\mu)}\,, \quad \beta,\mu>0\,, \quad z\in\mathbb{C}\,.$$

For information on the Mittag-Leffler-type functions the reader may consult *e.g.* [6,12].

Following [9] we invert the Fourier transforms of $K^{\theta\,(j)}_{\alpha,\beta}(x)$ by using the convolution theorem of the Mellin transforms, arriving at the Mellin-Barnes integral representation[c]:

$$K^{\theta\,(j)}_{\alpha,\beta}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})\,\Gamma(1-s)}{\Gamma(j-\frac{\beta}{\alpha}s)\,\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\, x^{\,s}\, ds\,, \qquad (2.9)$$

where $0 < \gamma < \min\{\alpha, 1\}$, and $\rho = (\alpha - \theta)/(2\,\alpha)$.

We note that the Mellin-Barnes integral representation (2.9)[d] allows us to construct computationally the fundamentals solutions of Eq. (1.2) for any triplet $\{\alpha, \beta, \theta\}$ by matching their convergent and asymptotic expansions, as shown in [9] for the first Green function. For the particular cases that allow simplifications in the integrand of Eq. (2.9), we obtain relevant expressions of the corresponding Green functions. This occurs in the following cases:
(a) for $j = 1$ and $\{0 < \alpha < 2,\, \beta = 1\}$ (*strictly space fractional diffusion*) where we have $K^{\theta\,(1)}_{\alpha,1}(x) = L^{\theta}_{\alpha}(x)$, *i.e.* the class of the strictly stable (non-Gaussian) densities[3] exhibiting fat tails (with the algebraic decay $\propto |x|^{-(\alpha+1)}$) and infinite variance;
(b) for $j = 1, 2$ and $\{\alpha = 2,\, 0 < \beta < 2\}$ (*time fractional diffusion* including *standard diffusion*), where we have $K^{0\,(j)}_{2,\beta}(x) = M^{(j)}_{\beta/2}(x)/2$, *i.e.* the class of the Wright-type densities [5,7,8,9,11] exhibiting stretched exponential tails and finite variance proportional to $t^{\beta+j-1}$;
(c) for $j = 1$ and $\{0 < \alpha = \beta < 2\}$ (*neutral fractional diffusion*), where we have $K^{\theta\,(1)}_{\alpha,\alpha}(x) = N^{\theta}_{\alpha}(x)$, *i.e.* the class of the Cauchy-type densities [9].

Based on the arguments outlined in[9], we extend the meaning of probability density to the cases $\{0 < \alpha < 2\,,\, 0 < \beta < 1\}$ and $\{1 < \beta \le \alpha < 2\}$ by proving the following composition rules of the Mellin convolution type:

$$K^{\theta\,(j)}_{\alpha,\beta}(x) = \begin{cases} \alpha \displaystyle\int_0^{\infty} \left[\xi^{\alpha-1}\, M^{(j)}_{\beta}\,(\xi^{\alpha})\right]\, L^{\theta}_{\alpha}\,(x/\xi)\, \frac{d\xi}{\xi}\,, & 0 < \beta < 1\,, \\ \displaystyle\int_0^{\infty} M^{(j)}_{\beta/\alpha}(\xi)\, N^{\theta}_{\alpha}(x/\xi)\, \frac{d\xi}{\xi}\,, & 0 < \beta/\alpha < 1\,. \end{cases} \qquad (2.10)$$

[c] The names refer to the two authors, who in the beginning of the past century developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, as revisited in[10], these integrals were first introduced in 1888 by S. Pincherle (Professor of Mathematics at the University of Bologna from 1880 to 1928).
[d] Readers acquainted with Fox H functions can recognize in (2.9) the representation of a certain function of this class, see *e.g.* [13]. Unfortunately, as far as we know, computing routines for this general class of special functions are not yet available.

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PREFACE

This volume contains the invited lectures and contributed papers presented at the XI International Conference on *Waves and Stability in Continuous Media* (WASCOM 2001) held June 3–9, 2001, in Porto Ercole (GR), Italy.

Ever since its initial edition organized in Catania 1981, the Conference aimed to bring together foreign and Italian researchers and scientists to discuss problems, promote collaborations and shape future directions for research in the field of stability and wave propagation in continuous media.

This cycle of conferences became a fixed meeting every two years: the further conferences have been held in Cosenza ('83), Bari ('85), Taormina ('87), Sorrento ('89), Acireale ('91), Bologna ('93), Palermo ('95), Monopoli ('97) and Vulcano ('99).

Every time the proceedings have been published, documenting the research work and progress in the area of waves and stability.

From a scientific point of view the success of this experience is confirmed by the fact that a remarkable group of italian researchers, from many different universities, has proposed several national projects in the field. The last project, entitled "Non Linear Mathematical Problems of Wave Propagation and Stability in Models of Continuous Media", co-ordinated by Prof. T. Ruggeri (Bologna), is the main proposer of the present conference.

The eleventh edition, the first of the third millennium, registered over 110 participants coming from more than 11 different countries. The topics covered by 29 main lectures and 52 short communications, within 10 sessions, were

- Discontinuity and shock waves
- Stability in Fluid Dynamics
- Small parameter problems
- Kinetic theories towards continuum models
- Non equilibrium thermodynamics
- Numerical applications

The Editors of the proceedings would like to thank the Scientific Committee who carefully suggested the invited lectures and selected the contributed papers, as well as the members of the Organizing Committee, coming from the Departments of Mathematics of the Universities of Napoli, Messina and Politecnico of Torino.

A special thank is addressed to all the participants to whom ultimately the success of the conference has to be ascribed.

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