

**The Legacy of A.Ya. Khintchine's  
Work in Probability Theory**

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# Contents

Preface	ix
Introduction	xi
CHAPTER 1	
First papers in Probability: 1924–1936	1
Appendix 1.1	
Khintchine A.Ya. On a general theorem related to the denumerable probabilities // C. R. Acad. Sci. Paris, 1 <sup>er</sup> semestre, Vol. 178, No. 75, 617-619 (1924) Séance du 11 Fevrier 1924	4
Appendix 1.2	
Khintchine A.Ya. On a theorem of Probability Theory // Fundamenta Mathematicae, 5, 9-20 (1924)	7
Appendix 1.3	
Khintchine A.Ya. On a new limit theorem of the Probability Theory // Mathematische Annalen, Vol. 101, 745-752 (1929)	19
Appendix 1.4	
Khintchine A.Ya. On the domain of attraction of the Gauss law // Giornale dell’Istituto Italiano degli Attuari, Vol. 6, No 4, 378-393 (1935)	27

## CHAPTER 2

The interaction with Paul Lévy 41

Appendix 2.1

Khintchine A.Ya. A new derivation of a formula by  
P. Lévy // Bulletin of Moscow State University  
(Buletten' Moscovskogo Gosudarstevnogo  
Universiteta) Vyp. 1, 1-5 (1937) 44

Appendix 2.2

Khintchine A.Ya. Invariant classes of distributions //  
Bulletin of Moscow State University (Buletten'  
Moscovskogo Gosudarstevnogo Universiteta) Vyp.  
5, 4-5 (1937) 50

Appendix 2.3

Khintchine A.Ya. Examples of random variables  
satisfying stable probability laws // Bulletin of  
Moscow State University (Buletten' Moscovskogo  
Gosudarstevnogo Universiteta) Vyp. 5, 6-9 (1937) 53

Appendix 2.4

Khintchine A.Ya. and Lévy P. On the stable laws //  
Comptes Rendus de l'Académie des Sciences, Paris,  
Vol. 202, 374-376 (1936) (Séance du 3 février 1936) 57

## CHAPTER 3

Infinitely divisible distributions 61

Appendix 3.1

Khintchine A.Ya. On the arithmetic of distributions  
// Bulletin of Moscow State University (Buletten'  
Moscovskogo Gosudarstevnogo Universiteta) Vyp.  
1, 6-17 (1937) 63

### Appendix 3.2

- Khintchine A.Ya. On a criterion for characteristic functions // Bulletin of Moscow State University (Bulleten' Moscovskogo Gosudarstvennogo Universiteta) Vyp. 5, 1-3 (1937) 75

### Appendix 3.3

- Khintchine A.Ya. To the theory of the infinitely divisible distributions // Math. Sbornik (Recueil Mathematique), Vol. 2(44), No. 1, 79-119 (1937) 79

## CHAPTER 4

- Khintchine's book on the distribution of the sum of independent random variables 127

### Appendix 4.1

- Khintchine A.Ya. Limit distributions for the sums of independent random variables // United Scientific-Technical Publishing Company, NKTP USSR, Editorial Board of Technical-Theoretical Literature, Moscow-Leningrad 1938 129

## CHAPTER 5

- Teaching Probability Theory and Analysis 231

## CHAPTER 6

- Last papers in Probability: 1950–1959 241

### Appendix 6.1

- Khintchine A.Ya. On sums of positive random variables // Doklady Acad. Sci. USSR, Vol. 71, No. 6, 1037-1039 (1950) 243

<b>Appendix 6.2</b>	
<b>Khintchine A.Ya. Limit theorem for sums of positive random variables // Ukranian Math. Journal (Ukrainskij matematicheskij zhurnal), Vol. 2, No. 4, 3-17 (1950)</b>	<b>247</b>
<b>References</b>	<b>261</b>
<b>Bibliography on A. Ya. Khintchine</b>	<b>265</b>
<b>Index</b>	<b>274</b>

# Preface

The Theory of Probability developed in 1920-1930s and progressed so rapidly that even nowadays some of the ideas and results of that period are re-established. It is very interesting from different points of view to understand how this branch of mathematics became one of the most important and applicable disciplines. In order to demonstrate this we can turn to the most influential results of that period concerning the main concepts of Probability Theory.

Among the contributors to the “great jump” in the development of Probability Theory in the above mentioned period, a special role belongs to Alexander Yakovlevich Khintchine. He obtained outstanding results, which form a very clear and rigorous style of handling important and profound problems. He created, together with Andrej Nikolaevich Kolmogorov, the well-known school of Probability Theory at Moscow State University. The work by Khintchine was highly regarded and evaluated in the context of the rapid development (in 1920s-1940s) of Probability Theory, in USSR as in Western Europe.

In this book we present the English translation of a number of relevant works by Khintchine supplemented by our comments and notes. In fact, the book has been motivated by the following reasons:

- Several important results by Khintchine were forgotten and were later re-discovered.
- A number of results were published in inaccessible places and were not available in English.
- The rigorous and clear style of Khintchine can help the readers to understand some recent results more easily.

We are grateful to many of our colleagues, especially those from Belarusian State University and Bologna University, who motivated the writing of this book. We would like to express our gratitude to Mrs. Janie Wardle for her kind attention to our work and for essential improving of its presentation. Special thanks to Professors Samuel Kotz, Rudolf Gorenflo, Ole Barndorff-Nielsen, Svetlozar Rachev and Marco Corazza for their encouragement. We are very much obliged to Doctor Alexander Titioura for the help in preparation of the manuscript.

This book was partly written during the stay of SR at the Department of Physics of Bologna University. We have appreciated the scientific atmosphere and perfect conditions for providing research at this Department.

Bologna-Minsk, March 2009



# Introduction

This book describes and highlights the work of the well-known Russian mathematician, Alexander Yakovlevich Khintchine<sup>1</sup> (1894–1959), who was one of the founders of the Russian (Soviet) mathematical school on Probability Theory.

A.Ya. Khintchine was born on July 19, 1894 in the village Kondrovo in the Kaluga region, about 150 km southwest of Moscow. From 1911 to 1916, he was a student of the Physical-Mathematical Faculty of Moscow State University (MSU), and indeed, during his scientific life, he was deeply involved and in contact with this University.

In the period of study at the University and in the first years of his research career, Khinchine was strongly influenced by the ideas and personality of N.N. Luzin. It is known that A.Ya. Khinchine presented his first scientific work at a meeting of the student mathematical club in November 1914 (see, *e.g.*, [17]).

The mathematical talent of this young outstanding student was noticed at the University by his teachers. After graduation at MSU, A.Ya. Khintchine was recommended to apply for professorship. His teaching career started in 1918 at Moscow Women's Polytechnical Institute. One year later he was invited to the Ivanovo-Voznesensk Polytechnical Institute, and soon after he became the dean of Physical-Mathematical Faculty of the newly founded Ivanovo-Voznesensk Pedagogical Institute. In 1922, the Research Institute of Mathematics and Mechanics was established at Moscow State University, A.Ya. Khintchine was invited to this Institute as a researcher. During a certain period he combined his research work in Moscow with lecturing at Ivanovo-Voznesensk. Finally, in 1927 he became a professor at the Moscow State University.

After his first significant publications A. Ya. Khintchine became known to the European probabilistic community. In 1928, he spent a couple of weeks at the University in Göttingen, one of the most important mathematical centers of the beginning of twentieth century. Here he wrote at least

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<sup>1</sup> Another transliteration of his name in English is Alexander Iacovlevich Khinchin.

two key papers, published in 1929 in *Mathematische Zeitschrift* and in *Mathematische Annalen* ([G 46] and [G 48], respectively).<sup>2</sup>

Khintchine was a member of the Soviet delegation at the International Congress of Mathematicians held in Bologna (Italy) in September 1928 (see, e.g., [33]). The Russian delegation was represented by 27 scientists including S. Bernstein (Kharkov), A.Ya. Khintchine (Moscow), V. Romanovsky (Tashkent) and E. Slutsky (Moscow). We note, however, that Khintchine did not present any communication so that he did not publish a paper in the Proceedings of the Congress (which appeared in 1929-1932).

From 1927 all A.Ya. Khintchine's subsequent scientific and teaching activity was connected with Moscow State University. He was the head of the chair of Probability Theory, then the head of the chair of Mathematical Analysis, and the director of the Research Institute of Mathematics and Mechanics at MSU. He passed away on November 18, 1959 after a long illness.

The aim of this book is to describe the role of the works of A. Ya. Khintchine in the early development of modern Probability Theory. The papers and the monographs by Khintchine were written not only in Russian, but also in German, in French and in Italian, which proves and emphasises his attitude to popularize his results and to interact with the international mathematical community. Nowadays there are only a few translations of his works into English. Therefore we considered that it was important to present the English translations of many of his works including his fundamental monograph on Limit Distributions for the Sum of Independent Random Variables published in 1938.

Our book has been preceded by the publication of the authors' recent paper concerning the origin of infinitely divisible distributions (now available on the Web <http://arXiv.org/math/arXiv:0801.1910>):

F. Mainardi and S. Rogosin, The origin of infinitely divisible distributions: from de Finetti's problem to Lévy-Khintchine formula, *Mathematical Methods for Economics and Finance*, Vol. 1 (2006), 37–55.

The book comprises an Introduction, six Chapters and a Bibliography.

In Chapter 1, we present the results of some of earlier papers by Khintchine. We can consider 1922 as the year in which he began his new research activity, which was mainly devoted to the development of Probability Theory. In spite of the wide spectrum of his scientific interests, this branch of Mathematics was the key for him. His earliest papers in Probability Theory appeared in 1924 and were highly evaluated and regarded by the well-known mathematicians E. Borel, W. Sierpinsky, H. Steinhaus, P. Lévy. During the 1920s and 1930s he had published 47 works on Probability and its applications, including remarkable monographs

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<sup>2</sup> Throughout our book the citation of the type [G . .] means the corresponding paper in the list of Khintchine publications, presented by B. V. Gnedenko in his article [17]. We have included the revised version of this list as an Appendix at the end of the book.

[G 65] and [G 92]. These monographs are recognized by the probabilistic community as classic books:

- A.Ya. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, 1933, pp. 77. Reprinted by Chelsea Publ., New-York, 1948.

- A.Ya. Khintchine, *Limit Distributions for the Sum of Independent Random Variables*. O.N.T.I., Moscow, 1938, pp. 115 (in Russian).

In Chapter 2, we present the results partly originated by the influence of Paul Lévy, one of the founders of the modern Theory of Probability, whom Khintchine had met at the Bologna International Congress of Mathematicians in 1928. This meeting can be considered as the initial point of the exchange of ideas between these two scientists, and their interaction for almost a decade.

In Chapter 3, we discuss the results by Khintchine on infinitely divisible distributions, partly using some material from our paper [33]. One of the most important results by Khintchine was the construction of a general theory of limit distributions for sums of independent random variables. His contributions to this topic were presented in his article [G 91] published in 1937 and in a wider and more detailed form in his monograph [G 92] published in 1938. The fundamental statement of his theory is the following: the class of limit distributions for sums of independent asymptotically negligible random variables coincides with the class of infinitely divisible distributions.

We pay special attention to this monograph [G 92], which is very characteristic for the whole work of Khintchine. Since his book exists only in Russian, we propose the translation of this monograph in Chapter 4.

Chapter 5 is devoted to the educational aspects of Khintchine's scientific life. He had a special interest in Mathematical Education at Universities, as well as in Secondary Schools. He published several papers and books on this topic. He tried to include in the range of his interests a larger and larger group of problems of Probability Theory and its applications. He attracted many young mathematicians to work on the solution of the problems, and in this way formed the Russian school of Probability Theory.

In Chapter 6, we present a few examples of the latest works by Khintchine. In this last period of his active scientific life he was interested in the foundation of Statistical Physics and in the development of some new branches of Mathematics, in particular Information Theory and Queueing Theory.

