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SPECIAL FUNCTIONS VIA MELLIN-BARNES INTEGRALS

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Mellin-Barnes integrals are characterised by integrals involving products and ratios of gamma functions with integration contours that thread their way around sequences of poles of the integrands. They are a powerful tool in the development of convergent or asymptotic expansions of functions defined by integrals, sums or differential equations. Furthermore they can be combined with the closely related Mellin transform. The great utility of these integrals resides in the facts that the asymptotic behaviour near the origin and at infinity of the function being represented is related to the singularity structure in the complex plane of the resulting integrand and to inherent flexibility associated with deformation of the contour of integration over subsets of these singularities. For an exhaustive treatment of the Mellin-Barnes integrals we refer to the recent monograph by Paris and Kaminski [5].

The names refer to the two authors, who in the first 1910's developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, as pointed out by Tricomi in [1] (Vol. 1, Ch. 1, §1.19, p. 49), these integrals were first used by S. Pincherle in 1888 [6]. For a revisited analysis of the pioneering work of Pincherle (1853-1936, Professor of Mathematics at the University of Bologna from 1880 to 1928) we refer to the recent paper by Mainardi and Pagnini [3].

The Mellin-Barnes integrals have recently been applied to numerically evaluate the fundamental solutions of space-time fractional differential equations for anomalous diffusion, see [2], [4]. These solutions turn out to be high transcendental functions belonging to the Fox H class. The most simple, not trivial, example of this class is the Mittag-Leffler function, that generalizes in a natural way the exponential: for negative argument it exhibits a power law decay, suitable to explain slow relaxation phenomena.

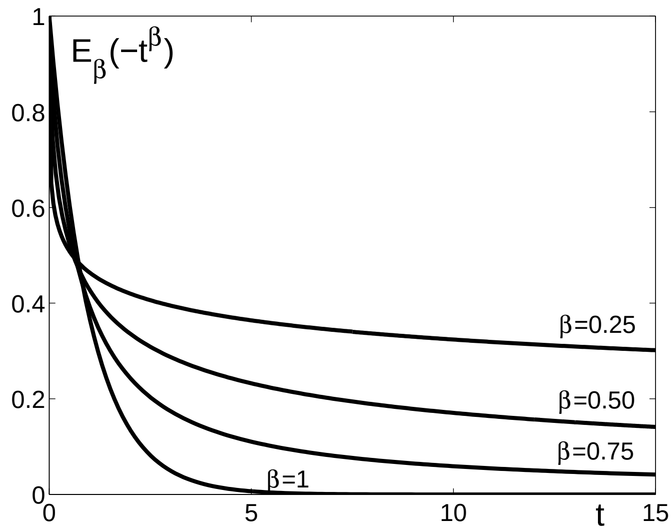

Figure 1: The Mittag-Leffler function $E_\beta(-t^\beta)$ for $\beta = 0.25, 0.50, 0.75, 1$.

References

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[4] G. Pagnini, *Generalized Equations for Anomalous Diffusion and their Fundamental Solutions*, Thesis for Degree in Physics, University of Bologna, October 2000, in Italian. [Supervisors: Prof. F. Mainardi and Prof. R. Gorenflo]

[5] R.B. Paris and D. Kaminski, *Asymptotic and Mellin-Barnes Integrals*, Cambridge Univ. Press, Cambridge (2001).

[6] S. Pincherle, Sulle funzioni ipergeometriche generalizzate, *Atti R. Accademia Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* (Ser. 4) **4** (1888) 694-700, 792-799.