INTERDISCIPLINARY WORKSHOP
"FROM WAVES TO DIFFUSION AND BEYOND"
BOLOGNA, Italy : December 20, 2002

RANDOM WALKS AND FRACTIONAL DIFFUSION

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A physical-mathematical approach to anomalous diffusion may be based on generalized diffusion equations (containing derivatives of fractional order in space or/and time) and related random walk models. By the space-time fractional diffusion equation we mean an evolution equation of the form

$$ {}_xD_\theta^\alpha\, u(x,t) \,=\, {}_tD_*^\beta\, u(x,t)\,, \quad x \in I\!R\,, \quad t \in I\!R^+\,, $$

obtained from the standard linear diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $\alpha \in (0,2]$ and skewness $\theta\,(|\theta| \le \min\{\alpha, 2-\alpha\})$, and the first-order time derivative with a Caputo derivative of order $\beta \in (0,1]$.

The fundamental solution (for the Cauchy problem) of the fractional diffusion equation can be interpreted as a spatial probability density evolving in time of a peculiar self-similar stochastic process. We view it as a generalized diffusion process, that we call *fractional diffusion process*. By adopting appropriate finite-difference schemes of solution, random walk models (discrete in space and time) have been generated, see *e.g.* [2, 5], which provide interesting realizations of the fractional diffusion processes.

However, a more general approach to anomalous diffusion is known to be provided by the master equation for a continuous time random walk (CTRW), formerly introduced by Montroll and Weiss in 1965 [4], where the wandering particle makes jumps at random times. CTRW is thus defined by a waiting time distribution and a jump width distribution, which are usually assumed to be independent of each other.

In [1] this master equation is shown to reduce to our fractional diffusion equation by a properly scaled passage to the limit of compressed waiting times and jump widths. Finally, following [3, 6], we describe a method of simulation and display (via graphics) results of a few numerical case studies, see *e.g.* Fig. 1.

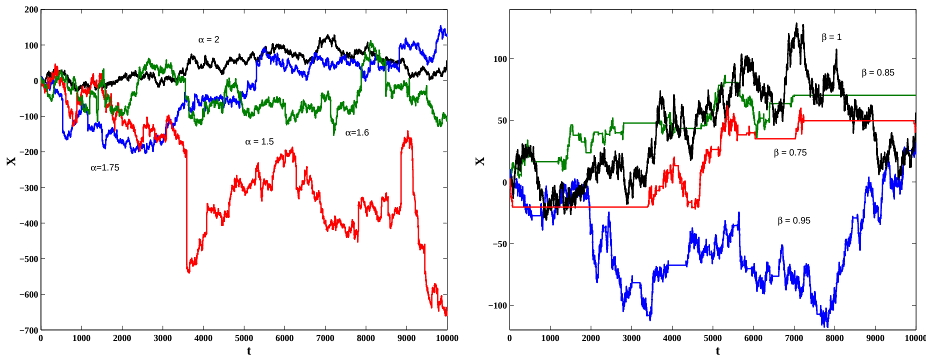

Figure 1: Sample paths for CTRW's with $\theta = 0$. At left we have: $\beta = 1$ and $\alpha = 2,\ 1.75,\ 1.6,\ 1.5$; at right we have: $\alpha = 2$ and $\beta = 1,\ 0.95,\ 0.85,\ 0.75$.

References

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[6] A. Vivoli, *Non-Gaussian Stochastic Processes and Their Applications*, Thesis for Degree in Physics, University of Bologna, March 2002, in Italian. [Supervisors: Prof. F. Mainardi and Prof. R. Gorenflo]