COMPUTATION OF THE MITTAG-LEFFLER FUNCTION $E_{\alpha,\beta}(z)$ AND ITS DERIVATIVE

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Dedicated to Francesco Mainardi,
Professor at the University of Bologna,
on the occasion of his 60-th birthday

Abstract

In this paper algorithms for numerical evaluation of the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \ \alpha > 0, \ \beta \in \mathbf{R}, \ z \in \mathbf{C}$$

and its derivative for all values of the parameters $\alpha > 0$, $\beta \in \mathbf{R}$ and all values of the argument $z \in \mathbf{C}$ are presented. For different parts of the complex plane different numerical techniques are used. In every case we provide estimates for accuracy of the computation; numerous pictures showing the behaviour of the Mittag-Leffler function for different values of the parameters and on different lines in the complex plane are included. The ideas und techniques employed in the paper can be used for numerical evaluation of other functions of the hypergeometric type. In particular, the same method with some small modifications can be applied for the Wright function which plays a very important role in the theory of partial differential equations of fractional order.

Mathematics Subject Classification: 33E12, 65D20, 33F05, 30E15
Key Words and Phrases: Mittag-Leffler function, integral representations of special functions, numerical computation of special functions, Mathematica programs

1. Introduction

The function

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)},\ \alpha>0,\ z\in\mathbf{C}, \tag{1}$$

was introduced by Mittag-Leffler in 1902 in connection with his method of summing divergent series. It is suited to serve as a very simple example of an entire function of a specified order $1/\alpha$ and of normal type. Further investigations of properties of this function and its applications to various questions of mathematical analysis have been carried out by Wiman [33]-[34] and Buhl [7]. An important generalization,

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\ \alpha>0,\ \beta\in\mathbf{C},\ z\in\mathbf{C}, \tag{2}$$

was introduced by Humbert [22], Agarwal [1], Humbert and Agarwal [23], Cleota and Hughes [9].

In recent years the interest in functions of Mittag-Leffler type among scientists, engineers and applications-oriented mathematicians has deepened. This interest is caused by the close connection of these functions to differential equations of fractional (meaning: non-integer) order and integral equations of Abel type, such equations becoming more and more popular in modelling natural and technical processes and situations (Babenko [2], Bagley and Torvik [3], Blair [4]-[5], Caputo and Mainardi [8], El-Sayed [12], Gorenflo et al. [16], Gorenflo and Mainardi [17]-[19], Gorenflo and Rutman [20], Gorenflo and Vessella [21], Luchko [24], Luchko and Srivastava [27], Mainardi [28], Schneider and Wyss [30], Slonimski [31], Westerlund and Ekstam [32]).

Although there already exists a rich literature on analytical methods for solving differential equations of fractional order, it is to be remarked that solutions in closed form have been found only for such equations with constant coefficients and for a rather small class of equations with particular variable coefficients. In general, numerical solution techniques are required.

In connection with the basic role of Mittag-Leffler type functions for the solution of fractional differential equations and integral equations of Abel type it seems important as a first step to develop their theory and stable methods for their numerical computation. The most simple function of Mittag-Leffler type, $E_\alpha(z)$, depends on two variables: the complex argument z and the real parameter α. The generalizations need at least one more argument, and α may be allowed to be complex. Experience in the computation of special functions of mathematical physics teach us that in distinct parts of the complex plane different numerical techniques should be used. The aim of this paper is the development of some methods for computing the Mittag-Leffler function (2) and its derivative as well as the assessment of the range of their applicability and accuracy. We propose Taylor series in the case $0 < \alpha \leq 1$ for small $|z|$, asymptotic representations for $|z|$ of large magnitude, and special integral representations for intermediate values of the argument z. By aid of a recursion formula we reduce the case $1 < \alpha$ to the case $0 < \alpha \leq 1$. We devise the needed algorithms in the system MATHEMATICA.

2. Integral representations of the Mittag-Leffler function

Integral representations play a prominent role in the analysis of entire functions. For the Mittag-Leffler function (2) such representations in form of an improper integral along the Hankel loop have been treated in the case $\beta = 1$ and in the general case with arbitrary β by Erdélyi et al. [13] and Dzherbashyan [10], [11]. They considered

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i\alpha}\int_{\gamma(\epsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\, d\zeta,\ z \in G^{(-)}(\epsilon;\delta), \tag{3}$$

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \frac{1}{2\pi i\alpha}\int_{\gamma(\epsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\, d\zeta,\ z \in G^{(+)}(\epsilon;\delta), \tag{4}$$

under the conditions

$$0 < \alpha < 2,\ \frac{\pi\alpha}{2} < \delta \leq \min\{\pi, \pi\alpha\}. \tag{5}$$

The contour $\gamma(\epsilon;\delta)$ can be seen in Figure 1. It consists of two rays $S_{-\delta}$ and S_δ ($\arg\zeta = -\delta$, $|\zeta| \geq \epsilon$ and $\arg\zeta = \delta$, $|\zeta| \geq \epsilon$) and a circular arc $C_\delta(0;\epsilon)$ ($|\zeta| = \epsilon$, $-\delta \leq \arg\zeta \leq \delta$). On its left side there is a region $G^{(-)}(\epsilon,\delta)$, on its right side a region $G^{(+)}(\epsilon;\delta)$.

Using the integral representations in (3), (4) it is not difficult to get asymptotic expansions for the Mittag-Leffler function in the complex plane. Let $0 < \alpha < 2$, β be an arbitrary number, and δ be chosen to satisfy the condition (5). Then we have, for any $p \in \mathbf{N}$ and $|z| \to \infty$:
Whenever $|\arg z| \leq \delta$,

$$E_{\alpha,\beta}(z) = \frac{z^{\frac{(1-\beta)}{\alpha}} e^{z^{1/\alpha}}}{\alpha} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}). \tag{6}$$

Whenever $\delta \leq |\arg z| \leq \pi$,

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}). \tag{7}$$

These formulas are also used in our numerical algorithm.

In what follows we restrict our attention to the case $\beta \in \mathbf{R}$, the most important in the applications. For the purpose of numerical computation we look for integral representations better suited than (3) und (4). With

$$\phi(\zeta, z) = \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}$$

we then have

$$I = \frac{1}{2\pi i\alpha}\int_{\gamma(\epsilon;\delta)} \phi(\zeta, z)\, d\zeta = \frac{1}{2\pi i\alpha}\int_{S_{-\delta}} \phi(\zeta, z)\, d\zeta \tag{8}$$

$$+\frac{1}{2\pi i\alpha}\int_{C_\delta(0;\epsilon)} \phi(\zeta, z)\, d\zeta + \frac{1}{2\pi i\alpha}\int_{S_\delta} \phi(\zeta, z)\, d\zeta = I_1 + I_2 + I_3.$$

Now we transform the integrals $I_1,\ I_2$ and I_3. For I_1 we take $\zeta = re^{-i\delta},\ \epsilon \le r < \infty$ and get

$$I_1 = \frac{1}{2\pi i\alpha}\int_{S_{-\delta}} \phi(\zeta, z)\, d\zeta = \frac{1}{2\pi i\alpha}\int_{+\infty}^{\epsilon} \frac{e^{(re^{-i\delta})^{1/\alpha}}(re^{-i\delta})^{(1-\beta)/\alpha}}{re^{-i\delta} - z}\, e^{-i\delta}\, dr. \tag{9}$$

Analogously for (with $\zeta = re^{i\delta},\ \epsilon \le r < \infty$)

$$I_3 = \frac{1}{2\pi i\alpha}\int_{S_\delta} \phi(\zeta, z)\, d\zeta = \frac{1}{2\pi i\alpha}\int_{\epsilon}^{+\infty} \frac{e^{(re^{i\delta})^{1/\alpha}}(re^{i\delta})^{(1-\beta)/\alpha}}{re^{i\delta} - z}\, e^{i\delta}\, dr. \tag{10}$$

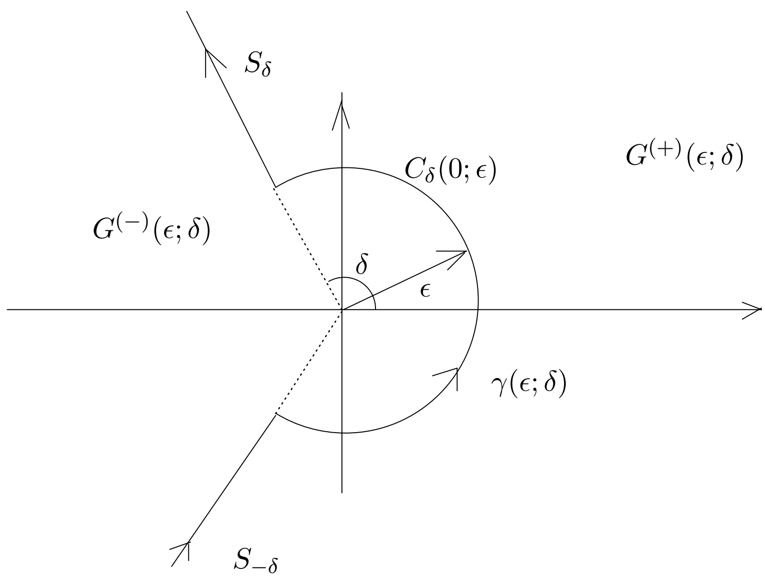

Figure 1: The contour $\gamma(\epsilon;\delta)$

For I_2 we have $\zeta = \epsilon e^{i\varphi},\ -\delta \leq \varphi \leq \delta$ and

$$I_2 = \frac{1}{2\pi i\alpha}\int_{C_\delta(0;\epsilon)} \phi(\zeta, z)\, d\zeta = \frac{1}{2\pi i\alpha}\int_{-\delta}^{\delta} \frac{e^{(\epsilon e^{i\varphi})^{1/\alpha}}(\epsilon e^{i\varphi})^{(1-\beta)/\alpha}}{\epsilon e^{i\varphi} - z}\, \epsilon i e^{i\varphi}\, d\varphi \tag{11}$$

$$= \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha}\int_{-\delta}^{\delta} \frac{e^{\epsilon^{1/\alpha}e^{i\varphi/\alpha}} e^{i\varphi((1-\beta)/\alpha+1)}}{\epsilon e^{i\varphi} - z}\, d\varphi = \int_{-\delta}^{\delta} P[\alpha, \beta, \epsilon, \varphi, z]\, d\varphi,$$

with

$$P[\alpha, \beta, \epsilon, \varphi, z] = \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha}\frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\omega) + i\sin(\omega))}{\epsilon e^{i\varphi} - z}, \tag{12}$$

$$\omega = \epsilon^{1/\alpha}\sin(\varphi/\alpha) + \varphi(1 + (1-\beta)/\alpha).$$

We rewrite the sum $I_1 + I_3$ as

$$I_1 + I_3 = \frac{1}{2\pi i\alpha}\int_{\epsilon}^{+\infty} \left\{\frac{e^{(re^{i\delta})^{1/\alpha}}(re^{i\delta})^{(1-\beta)/\alpha}}{re^{i\delta} - z}\, e^{i\delta}\right.$$

$$\left. - \frac{e^{(re^{-i\delta})^{1/\alpha}}(re^{-i\delta})^{(1-\beta)/\alpha}}{re^{-i\delta} - z}\, e^{-i\delta}\right\} dr$$

$$= \frac{1}{2\pi i\alpha}\int_{\epsilon}^{+\infty} r^{(1-\beta)/\alpha} e^{r^{1/\alpha}\cos(\delta/\alpha)}\left\{\frac{e^{i(r^{1/\alpha}\sin(\delta/\alpha)+\delta(1+(1-\beta)/\alpha))}}{re^{i\delta} - z}\right.$$

$$\left. - \frac{e^{-i(r^{1/\alpha}\sin(\delta/\alpha)+\delta(1+(1-\beta)/\alpha))}}{re^{-i\delta} - z}\right\} dr = \int_{\epsilon}^{+\infty} K[\alpha, \beta, \delta, r, z]\, dr,$$

with

$$K[\alpha, \beta, \delta, r, z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{r^{1/\alpha}\cos(\delta/\alpha)}\frac{r\sin(\psi - \delta) - z\sin(\psi)}{r^2 - 2rz\cos(\delta) + z^2}, \tag{13}$$

$$\psi = r^{1/\alpha}\sin(\delta/\alpha) + \delta(1 + (1-\beta)/\alpha).$$

By aid of (8)-(13) we can rewrite the formulas (3) and (4) as

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{+\infty} K[\alpha, \beta, \delta, r, z]\, dr + \int_{-\delta}^{\delta} P[\alpha, \beta, \epsilon, \varphi, z]\, d\varphi,\ \ z \in G^{(-)}(\epsilon; \delta), \tag{14}$$

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{+\infty} K[\alpha, \beta, \delta, r, z]\, dr + \int_{-\delta}^{\delta} P[\alpha, \beta, \epsilon, \varphi, z]\, d\varphi \tag{15}$$

$$+ \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}},\ \ z \in G^{(+)}(\epsilon; \delta).$$

Let us now consider the case $0 < \alpha \leq 1,\ z \neq 0$. By condition (5) we can choose $\delta = \min\{\pi, \pi\alpha\} = \pi\alpha$. Then the kernel function (13) looks simpler:

$$K[\alpha, \beta, \pi\alpha, r, z] = \tilde{K}[\alpha, \beta, r, z] \tag{16}$$

$$= \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2}.$$

In the formulas (14)-(16) for computation of the function $E_{\alpha,\beta}(z)$ at the arbitrary point $z \in \mathbf{C},\ z \neq 0$ we will distinguish three possibilities for arg z, namely
A) $|\arg z| > \pi\alpha$,
B) $|\arg z| = \pi\alpha$,
C) $|\arg z| < \pi\alpha$.

First we conside the case A): $|\arg z| > \pi\alpha$.

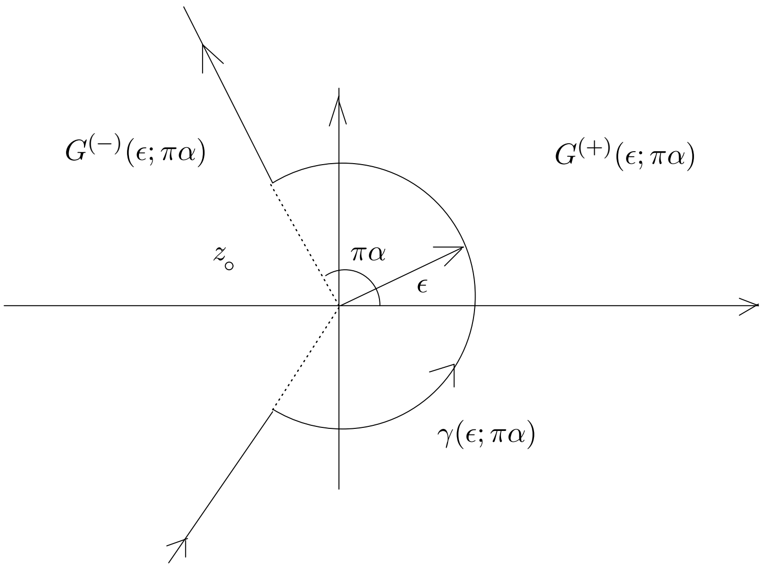

Figure 2: The case $|\arg z| > \pi\alpha$

In this case z always (for arbitrary ϵ) lies in the region $G^{(-)}(\epsilon;\delta)$ (see Figure 2) and we arrive at

THEOREM 2.1. *Under the conditions*

$$0 < \alpha \leq 1,\ \beta \in \mathbf{R},\ |\arg z| > \pi\alpha,\ z \neq 0$$

the function $E_{\alpha,\beta}(z)$ *has the representations*

$$E_{\alpha,\beta}(z) = \int_\epsilon^\infty \tilde{K}[\alpha,\beta,r,z]\,dr + \int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]\,d\varphi,\ \epsilon > 0,\quad \beta \in \mathbf{R}, \tag{17}$$

$$E_{\alpha,\beta}(z) = \int_0^\infty \tilde{K}[\alpha,\beta,r,z]\,dr, \quad \textit{if}\ \ \beta < 1+\alpha, \tag{18}$$

$$E_{\alpha,\beta}(z) = -\frac{\sin(\pi\alpha)}{\pi\alpha}\int_0^\infty \frac{e^{-r^{1/\alpha}}}{r^2 - 2rz\cos(\pi\alpha) + z^2}\,dr - \frac{1}{z}, \quad \textit{if } \beta = 1+\alpha \tag{19}$$

with

$$\tilde{K}[\alpha,\beta,r,z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2},$$

$$P[\alpha,\beta,\epsilon,\varphi,z] = \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha} \frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\omega) + i\sin(\omega))}{\epsilon e^{i\varphi} - z},$$

$$\omega = \epsilon^{1/\alpha}\sin(\varphi/\alpha) + \varphi(1 + (1-\beta)/\alpha).$$

P r o o f. The representation (17) follows immediately from (14) with $\delta = \pi\alpha$ and (16) and holds for arbitrary $\beta \in \mathbf{R}$. Let us now consider (17) for $\beta < 1+\alpha$. In this case we have

$$|P[\alpha,\beta,\epsilon,\varphi,z]| \le C\epsilon^{1+(1-\beta)/\alpha}, \ -\pi\alpha \le \varphi \le \pi\alpha, \ 0 < \epsilon < 1$$

with a constant C not depending on φ. Consequently,

$$\int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]\,d\varphi \ \to 0 \ \text{ if } \ \epsilon \to 0. \tag{20}$$

Further we have

$$\tilde{K}[\alpha,\beta,r,z] = O(r^{(1-\beta)/\alpha}), \ r \to 0.$$

This means: If $\beta < 1+\alpha$ the integral

$$\int_0^\infty \tilde{K}[\alpha,\beta,r,z]\,dr$$

is a convergent improper integral with singularity at $r = 0$, but it is there not singular if $\beta \le 1$. Using (20), we can calculate a limit for $\epsilon \to 0$ in the representation (17) and so arrive at the representation (18).

Finally, in the case $\beta = 1+\alpha$ we have the formulas

$$P[\alpha,\beta,\epsilon,\varphi,z] = \frac{1}{2\pi\alpha}\frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\epsilon^{1/\alpha}\sin(\varphi/\alpha)) + i\sin(\epsilon^{1/\alpha}\sin(\varphi/\alpha)))}{\epsilon e^{i\varphi} - z},$$

$$\lim_{\epsilon\to 0}\int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]\,d\varphi = \int_{-\pi\alpha}^{\pi\alpha}\lim_{\epsilon\to 0} P[\alpha,\beta,\epsilon,\varphi,z]\,d\varphi \tag{21}$$
$$= \int_{-\pi\alpha}^{\pi\alpha}\left(-\frac{1}{2\pi\alpha z}\right)d\varphi = -\frac{1}{z},$$

the corresponding integral converging uniformly with respect to ϵ. Hence for $\beta = 1+\alpha$ there holds the formula

$$\tilde{K}[\alpha,\beta,r,z] = -\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{e^{-r^{1/\alpha}}}{r^2 - 2rz\cos(\pi\alpha) + z^2} \tag{22}$$

and the integral

$$\int_0^\infty \tilde{K}[\alpha, \beta, r, z]\, dr$$

is non-singular in the point $r = 0$. The representation (19) now follows from (17), (21) und (22). ■

Example: Let $\beta = 1$, $0 < \alpha < 1$, $z = -t^\alpha < 0$.
Theorem 2.1 then yields the representation

$$E_{\alpha,1}(-t^\alpha) = \int_0^\infty \frac{1}{\pi\alpha} e^{-r^{1/\alpha}} \frac{t^\alpha \sin(\pi\alpha)}{r^2 + 2rt^\alpha \cos(\pi\alpha) + t^{2\alpha}}\, dr,$$

which by insertion of $r = x^\alpha t^\alpha$ can be transformed to

$$E_{\alpha,1}(-t^\alpha) = \int_0^\infty \frac{1}{\pi} e^{-xt} \frac{x^{\alpha-1} \sin(\pi\alpha)}{x^{2\alpha} + 2x^\alpha \cos(\pi\alpha) + 1}\, dx. \tag{23}$$

For the representation (23) which can be obtained by the Laplace transform method see Gorenflo and Mainardi [17].

The next case we consider is the case B): $|\arg z| = \pi\alpha$.

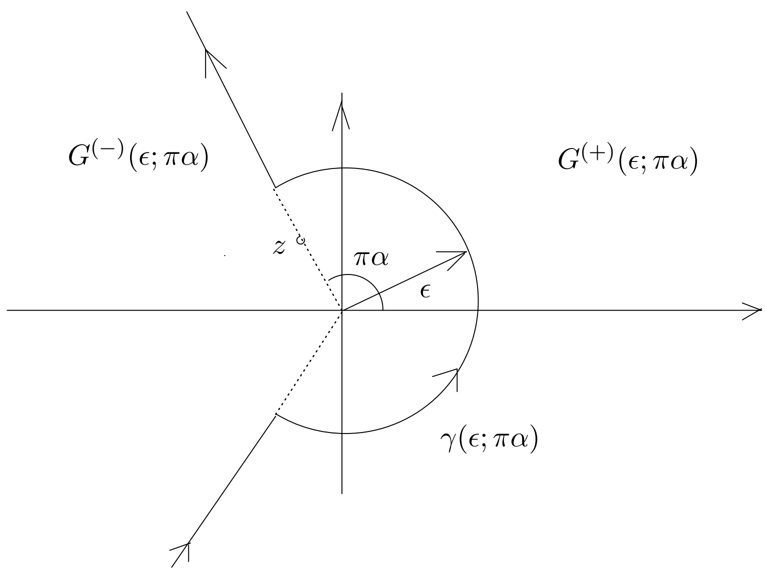

Figure 3: The case $|\arg z| = \pi\alpha$

In this case we are not allowed to make ϵ arbitrarily small in our contour $\gamma(\epsilon; \delta)$, because z is directly lying on this contour if $\epsilon < |z|$ (see Figure 3).

THEOREM 2.2. *Under the conditions*

$$0 < \alpha \leq 1,\ \beta \in \mathbf{R},\ |\arg z| = \pi\alpha,\ z \neq 0$$

the function $E_{\alpha,\beta}(z)$ *has the representation*

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} \tilde{K}[\alpha,\beta,r,z]\,dr + \int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]\,d\varphi,\ \ \epsilon > |z|, \tag{24}$$

with

$$\tilde{K}[\alpha,\beta,r,z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2},$$

$$P[\alpha,\beta,\epsilon,\varphi,z] = \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha} \frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\omega) + i\sin(\omega))}{\epsilon e^{i\varphi} - z},$$

$$\omega = \epsilon^{1/\alpha}\sin(\varphi/\alpha) + \varphi(1 + (1-\beta)/\alpha).$$

P r o o f. If $\epsilon > |z|$ then z is lying in the region $G^{(-)}(\epsilon;\delta)$ and the representation (24) follows from (14) with $\delta = \pi\alpha$ and (16). ■

Finally let us discuss the case C): $|\arg z| < \pi\alpha$.

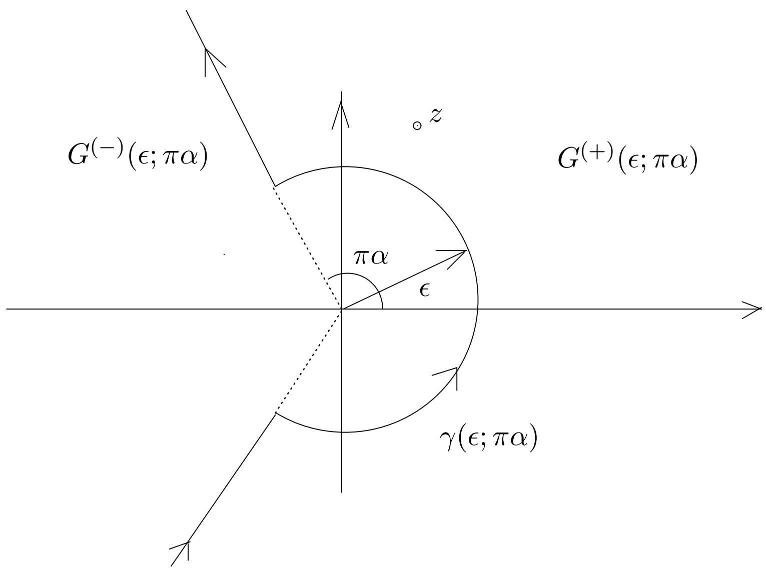

Figure 4: The case $|\arg z| < \pi\alpha$

In this case, if $0 < \epsilon < |z|$, then z is in the region $G^{(+)}(\epsilon;\delta)$, and we have

THEOREM 2.3. *Under the conditions*

$$0 < \alpha \le 1,\ \ \beta \in \mathbf{R},\ \ |\arg z| < \pi\alpha,\ \ z \ne 0$$

the function $E_{\alpha,\beta}(z)$ *possesses the representations*

$$E_{\alpha,\beta}(z) = \int_{\epsilon}^{\infty} \tilde{K}[\alpha,\beta,r,z]\,dr + \int_{-\pi\alpha}^{\pi\alpha} P[\alpha,\beta,\epsilon,\varphi,z]\,d\varphi \qquad (25)$$

$$+\frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}},\ \ 0 < \epsilon < |z|,\ \ \beta \in \mathbf{R},$$

$$E_{\alpha,\beta}(z) = \int_{0}^{\infty} \tilde{K}[\alpha,\beta,r,z]\,dr + \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}},\quad \text{if}\ \ \beta < 1+\alpha, \qquad (26)$$

$$E_{\alpha,\beta}(z) = -\frac{\sin(\pi\alpha)}{\pi\alpha}\int_{0}^{\infty} \frac{e^{-r^{1/\alpha}}}{r^2 - 2rz\cos(\pi\alpha) + z^2}\,dr \qquad (27)$$

$$-\frac{1}{z} + \frac{1}{\alpha z} e^{z^{1/\alpha}},\quad \text{if}\ \ \beta = 1+\alpha$$

with

$$\tilde{K}[\alpha,\beta,r,z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2},$$

$$P[\alpha,\beta,\epsilon,\varphi,z] = \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha} \frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\omega) + i\sin(\omega))}{\epsilon e^{i\varphi} - z},$$

$$\omega = \epsilon^{1/\alpha}\sin(\varphi/\alpha) + \varphi(1 + (1-\beta)/\alpha).$$

P r o o f. Similar to the proof of Theorem 2.1; use (15) instead of (14).

3. Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$

We have proved that for arbitrary $z \ne 0$ and $0 < \alpha \le 1$ the Mittag- Leffler function $E_{\alpha,\beta}(z)$ can be represented by one of the formulas (17)- (19), (24)-(26). We intend to use these formulas for numerical computation if $q < |z|,\ 0 < q < 1$ and $0 < \alpha \le 1$. In the case $|z| \le q,\ 0 < q < 1$ we compute $E_{\alpha,\beta}(z)$ for arbitrary $\alpha > 0$ by aid of the power series (2). The case $1 < \alpha$ can be reduced to the case $0 < \alpha \le 1$ by aid of a recursion formula. For computation of the function $E_{\alpha,\beta}(z)$ for arbitrary $z \in \mathbf{C}$ with arbitrary indices $\alpha > 0,\ \beta \in \mathbf{R}$, we will distinguish three possibilities:

A) $|z| \le q,\ 0 < q < 1$ (q is a fixed number), $0 < \alpha$,

B) $|z| > q,\ 0 < \alpha \le 1$ and

C) $|z| > q,\ 1 < \alpha$.

In each case we compute the Mittag-Leffler function with the prescribed accuracy $\rho > 0$.

First we consider the case A): $|z| \le q,\ 0 < q < 1$.

THEOREM 3.1. *In the case $|z| \le q$, $0 < q < 1$, $0 < \alpha$ the Mittag-Leffler function can be computed with the prescribed accuracy $\rho > 0$ by use of the formula*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{k_0} \frac{z^k}{\Gamma(\beta + \alpha k)} + \mu(z), \ |\mu(z)| \le \rho, \tag{28}$$

$$k_0 = \max\{[(1-\beta)/\alpha] + 1, \ [\ln(\rho(1-|z|))/\ln(|z|)]\}.$$

P r o o f. We transcribe the definition (2) into the form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{m} \frac{z^k}{\Gamma(\beta + \alpha k)} + \mu(z,m), \ \mu(z,m) = \sum_{k=m+1}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}.$$

For all $k \ge k_0 \ge [(1-\beta)/\alpha] + 1$ we have the inequality

$$\Gamma(\beta + \alpha k) \ge 1$$

and thus the estimate $(m + 1 \ge k_0)$

$$|\mu(z,m)| = \left| \sum_{k=m+1}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)} \right| \le \sum_{k=m+1}^{\infty} |z|^k = |z|^{m+1} \frac{1}{1-|z|}.$$

The estimate $|\mu(z)| := |\mu(z,k_0)| < \rho$ follows from the inequality $k_0 \ge [\ln(\rho(1-|z|))/\ln(|z|)]$. ■

REMARK 3.1. For computation of the function $E_{\alpha,\beta}(z)$ in case A) it is recommendable to choose in Theorem 3.1 a number q not very close to 1 (in order to have not a large number of terms in the sum of formula (28)). In our computer programs we have taken $q = 0.9$.

We proceed with the case B): $q < |z|$, $0 < \alpha \le 1$.

In this case we use the integral representations (17)-(19), (24)-(26). We then must compute numerically either the improper integral

$$I = \int_a^{\infty} \tilde{K}[\alpha, \beta, r, z]\, dr, \ a \in \{0, \epsilon\},$$

$$\tilde{K}[\alpha, \beta, r, z] = \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{r\sin(\pi(1-\beta)) - z\sin(\pi(1-\beta+\alpha))}{r^2 - 2rz\cos(\pi\alpha) + z^2},$$

or even the integral

$$J = \int_{-\pi\alpha}^{\pi\alpha} P[\alpha, \beta, \epsilon, \varphi, z]\, d\varphi, \ \epsilon > 0,$$

$$P[\alpha, \beta, \epsilon, \varphi, z] = \frac{\epsilon^{1+(1-\beta)/\alpha}}{2\pi\alpha} \frac{e^{\epsilon^{1/\alpha}\cos(\varphi/\alpha)}(\cos(\omega) + i\sin(\omega))}{\epsilon e^{i\varphi} - z},$$

$$\omega = \epsilon^{1/\alpha}\sin(\varphi/\alpha) + \varphi(1 + (1-\beta)/\alpha).$$

The second integral J (the integrand $P[\alpha,\beta,\epsilon,\varphi,z]$ being bounded and the limits of integration being finite) can be calculated with prescribed accuracy $\rho > 0$ by one of many product quadrature methods.

For calculating the first (improper) integral I over the bounded function $\tilde{K}[\alpha,\beta,r,z]$ we use

THEOREM 3.2. *The representation*

$$I = \int_a^\infty \tilde{K}[\alpha,\beta,r,z]\,dr = \int_a^{r_0} \tilde{K}[\alpha,\beta,r,z]\,dr\ + \mu(z),\ |\mu(z)| \le \rho,\ a \in \{0,\epsilon\} \quad (29)$$

is valid under the conditions

$$0 < \alpha \le 1,\ 0 < q < |z|,$$

$$r_0 = \begin{cases} \max\{1, 2|z|, (-\ln(\pi\rho/6))^\alpha\}, & \textit{if } \beta \ge 0,\\ \max\{(|\beta|+1)^\alpha, 2|z|, (-2\ln(\pi\rho/(6(|\beta|+2)(2|\beta|)^{|\beta|}))^\alpha\}, & \textit{if } \beta < 0. \end{cases}$$

P r o o f. We try to estimate the absolute value of

$$\mu(z,R) = \int_R^\infty \tilde{K}[\alpha,\beta,r,z]\,dr$$

in the expression

$$I = \int_a^R \tilde{K}[\alpha,\beta,r,z]\,dr + \mu(z,R),\ a \in \{0,\epsilon\}.$$

We first consider an estimate of the function $\tilde{K}[\alpha,\beta,r,z]$. Let $z_0 = e^{i\pi\alpha}$ and $r \ge 2|z|$. We then have

$$\frac{1}{|r^2 - 2rz\cos(\pi\alpha) + z^2|} = \frac{1}{r^2\left|\frac{z}{r} - z_0\right|\left|\frac{z}{r} - \overline{z}_0\right|}$$

$$\le \frac{1}{r^2\left(\left|\frac{z}{r}\right| - |z_0|\right)\left(\left|\frac{z}{r}\right| - |\overline{z_0}|\right)} = \frac{1}{r^2\left(1 - \left|\frac{z}{r}\right|\right)^2} \le \frac{4}{r^2}$$

and hence also

$$|\tilde{K}[\alpha,\beta,r,z]| \le \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{|r| + |z|}{|r^2 - 2rz\cos(\pi\alpha) + z^2|}$$

$$\le \frac{1}{\pi\alpha} r^{(1-\beta)/\alpha} e^{-r^{1/\alpha}} \frac{6r}{r^2} = \frac{6}{\pi\alpha} r^{(1-\beta)/\alpha - 1} e^{-r^{1/\alpha}}.$$

Hence we have, for $R \ge 2|z|$, the estimate

$$|\mu(z,R)| \le \int_R^\infty |\tilde{K}[\alpha,\beta,r,z]|\,dr \quad (30)$$

$$\leq \int_R^\infty \frac{6}{\pi\alpha} r^{(1-\beta)/\alpha-1} e^{-r^{1/\alpha}}\, dr = \frac{6}{\pi}\int_{R^{1/\alpha}}^\infty t^{-\beta} e^{-t} dt = \frac{6}{\pi}\Gamma(1-\beta, R^{1/\alpha}),$$

with the incomplete gamma function (see Erdélyi et al. [13])

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1}\, dt,\ \ x > 0$$

More estimates can be obtained by aid of

LEMMA 3.1. *For the incomplete gamma function* $\Gamma(1-\beta, x)$ *the following estimates hold:*

$$|\Gamma(1-\beta, x)| \leq e^{-x},\ \ x \geq 1,\ \ \beta \geq 0, \tag{31}$$

$$|\Gamma(1-\beta), x) \leq (|\beta| + 2) x^{-\beta} e^{-x},\ \ x \geq |\beta| + 1,\ \ \beta < 0. \tag{32}$$

P r o o f. For $t \geq x \geq 1$ and $\beta \geq 0$ we have the inequality $t^{-\beta} e^{-t} \leq e^{-t}$ and hence also (31) from

$$|\Gamma(1-\beta, x)| \leq \int_x^\infty e^{-t}\, dt = e^{-x}.$$

If $\beta < 0$ we determine $n \in \mathbf{N}$ such that $-(n+1) \leq \beta < -n$ and treat the integral $\Gamma(1-\beta, x)$ by n-fold partial integration:

$$\Gamma(1-\beta, x) = x^{-\beta} e^{-x} - \beta x^{-\beta-1} e^{-x} + \beta(\beta+1) x^{-\beta-2} e^{-x} \tag{33}$$

$$+ \ldots + (-1)^n \prod_{j=0}^{n-1} (\beta + j) x^{-\beta-n} e^{-x} + (-1)^{n+1} \int_x^\infty \prod_{j=0}^{n} (\beta+j) t^{-\beta-n-1} e^{-t}\, dt.$$

Now $\beta + n + 1 \geq 0$ and $x \geq |\beta| + 1 \geq 1$ in the last integral, and we can use the inequality (31):

$$\int_x^\infty t^{-\beta-n-1} e^{-t}\, dt \leq e^{-x}.$$

For $x \geq |\beta| + 1$ and $-(n+1) \leq \beta < -n$ we have

$$x^{-\beta} > |\beta x^{-\beta-1}| > |\beta(\beta+1) x^{-\beta-2}| > \ldots > |\beta(\beta+1)\ldots(\beta+n)|.$$

The latter inequalities together with the representation(33) yield the estimate

$$|\Gamma(1-\beta, x)| \leq (n+2) x^{-\beta} e^{-x} \leq (|\beta| + 2) x^{-\beta} e^{-x},\ \ |x| \geq |\beta| + 1,$$

and thus Lemma 3.1 is proved.

Let us return to the proof of Theorem 3.2. (30) and Lemma 3.1 yield the estimate

$$|\mu(z, R)| \leq \begin{cases} \frac{6}{\pi} e^{-R^{1/\alpha}}, & \beta \geq 0,\ \ R \geq 1 \\ \frac{6}{\pi}(|\beta| + 2) R^{-\beta/\alpha} e^{-R^{1/\alpha}}, & \beta < 0,\ \ R^{1/\alpha} \geq |\beta| + 1. \end{cases} \tag{34}$$

For $\beta \geq 0$ we have $|\mu(z)| := |\mu(z, r_0)| < \rho$, if $r_0 = \max\{1, 2|z|,\ (-\ln(\pi\rho/6))^\alpha\}$, and for $\beta < 0$ we should find an r_0 such that the inequality

$$\frac{6}{\pi}(|\beta| + 2) r_0^{-\beta/\alpha} e^{-r_0^{1/\alpha}} \leq \rho \tag{35}$$

is fulfilled. We solve this exercise by aid of

LEMMA 3.2. *For arbitrary $x, y, q > 0$ there holds the inequality*

$$x^y \leq (qy)^y e^{x/q}. \tag{36}$$

P r o o f. With $x = ay,\ a > 0$, the inequality (36) can be rewritten in the form $(ay)^y \leq (qy)^y e^{ay/q}$ which is equivalent to $a^y \leq q^y e^{ay/q}$ and hence also to

$$\left(\frac{a}{q}\right)^y \leq \left(e^{a/q}\right)^y.$$

The latter inequality is true because of

$$e^{a/q} \geq \max\{a/q, 1\},\ a, q > 0.$$

Hence Lemma 3.2 is proved.

Denoting $r_0^{1/\alpha}$ by x, $-\beta$ by y, and taking $q = 2$ we rewrite the inequality (35) by aid of (36) in the form

$$\frac{6}{\pi}(|\beta| + 2) r_0^{-\beta/\alpha} e^{-r_0^{1/\alpha}} = \frac{6}{\pi}(|\beta| + 2) x^y e^{-x}$$

$$\leq \frac{6}{\pi}(|\beta| + 2)(2y)^y e^{x/2} e^{-x} = \frac{6}{\pi}(|\beta| + 2)(2|\beta|)^{|\beta|} e^{-r_0^{1/\alpha}/2} < \rho.$$

Solving for r_0 we find $|\mu(z)| := |\mu(z, r_0)| \leq \rho$ for $\beta < 0$ and $r_0 = \max\{(|\beta| + 1)^\alpha, 2|z|,\ (-2\ln(\pi\rho/(6(|\beta| + 2)(2|\beta|)^{|\beta|}))^\alpha\}$. ■

The last case we have to discuss is the case C): $q < |z|,\ 1 < \alpha$. Here we use the recursion formula (see Dzherbashyan [11])

$$E_{\alpha,\beta}(z) = \frac{1}{m} \sum_{h=0}^{m-1} E_{\alpha/m,\beta}(z^{1/m} e^{i2\pi h/m})\ (m \geq 1). \tag{37}$$

In order to reduce case C) to the cases B) and A) we take $m = \lfloor \alpha \rfloor + 1$ in formula (37). Then $0 < \alpha/m \leq 1$, and we calculate the functions $E_{\alpha/m,\beta}(z^{1/m} e^{2\pi i h/m})$ as in case A) if $|z|^{1/m} \leq q < 1$, and as in case B) if $|z|^{1/m} > q$.

Remark 3.2. The ideas und techniques employed for the Mittag-Leffler function can be used for numerical calculation of other functions of the hypergeometric type. In particular, the same method with some small modifications can be applied for the Wright function playing a very important role in the theory of partial differential equations of fractional order (see for example Buckwar and Luchko [6], Gorenflo et al. [15], Luchko [25], Luchko and Gorenflo [26], Mainardi et al. [29]). To this end, the following representations (see Gorenflo et al. [14]) can be used instead of the representations (2), (3), and (4):

$$\phi(\rho,\beta;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k+\beta)},\ \rho > -1,\ \beta \in \mathbf{C},$$

$$\phi(\rho,\beta;z) = \frac{1}{2\pi i}\int_{\mathrm{Ha}} e^{\zeta + z\zeta^{-\rho}}\zeta^{-\beta}\,d\zeta,\ \rho > -1,\ \beta \in \mathbf{C},$$

where Ha denotes the Hankel path in the ζ-plane with a cut along the negative real semi-axis $\arg\zeta = \pi$.

4. Computation of the derivative of the function $E_{\alpha,\beta}(z)$

In many questions of analysis the derivative of a function plays an important role. The derivative of the Mittag-Leffler function can be used for example in iterative methods for determination of its zeros in the complex plane. The function $E_{\alpha,\beta}(z)$ being an entire function, we find by term-wise differentiation of its power series (2) the representation

$$E'_{\alpha,\beta}(z) = \sum_{k=1}^{\infty} \frac{kz^{k-1}}{\Gamma(\beta+\alpha k)} = \sum_{k=0}^{\infty} \frac{(k+1)z^k}{\Gamma(\alpha+\beta+\alpha k)}. \tag{38}$$

For numerical calculation of the function $E'_{\alpha,\beta}(z)$ with prescribed accuracy ρ we distinguish two cases: A) $|z| \le q < 1$ (q is a fixed number) and B) $|z| > q$.

We start with the case A): $|z| \le q < 1$.

Theorem 4.1. *In the case $|z| \le q < 1$ the derivative (38) of the Mittag-Leffler function can be calculated by aid of the formula*

$$E'_{\alpha,\beta}(z) = \sum_{k=0}^{k_0} \frac{(k+1)z^k}{\Gamma(\alpha+\beta+\alpha k)} + \mu(z),\ |\mu(z)| \le \rho, \tag{39}$$

$$k_0 = \max\{k_1,\ [\ln(\rho(1-|z|))/\ln(|z|)]\},$$

$$k_1 = \begin{cases} [(2-\alpha-\beta)/(\alpha-1)]+1, & \alpha > 1, \\ [(3-\alpha-\beta)/\alpha]+1, & 0 < \alpha \le 1,\ D \le 0, \\ \max\{\left[\frac{3-\alpha-\beta}{\alpha}\right]+1, \left[\frac{1-2\omega\alpha+\sqrt{D}}{2\alpha^2}\right]+1\}, & 0 < \alpha \le 1,\ D > 0, \end{cases} \tag{40}$$

$$\omega = \alpha + \beta - \frac{3}{2}, \ D = \alpha^2 - 4\alpha\beta + 6\alpha + 1$$

with prescribed accuracy $\rho > 0$.

P r o o f. By formula (38) we have

$$E'_{\alpha,\beta}(z) = \sum_{k=0}^{m} \frac{(k+1)z^k}{\Gamma(\alpha+\beta+\alpha k)} + \mu(z,m)$$

with

$$\mu(z,m) = \sum_{k=m+1}^{\infty} \frac{(k+1)z^k}{\Gamma(\alpha+\beta+\alpha k)}$$

and can estimate the absolute value of $\mu(z,m)$. Now $\Gamma(x) \geq 1$ for $x \geq 1$. So, we have the inequalities

$$\Gamma(\alpha+\beta+\alpha k) = (\alpha+\beta+\alpha k-1)\Gamma(\alpha+\beta+\alpha k-1) \geq \alpha+\beta+\alpha k-1 \geq k+1 \quad (41)$$

if

$$\alpha > 1, \ k \geq k_1 = [(2-\alpha-\beta)/(\alpha-1)] = 1,$$

$$\Gamma(\alpha+\beta+\alpha k) = (\alpha+\beta+\alpha k-1)(\alpha+\beta+\alpha k-2)\Gamma(\alpha+\beta+\alpha k-2) \quad (42)$$
$$\geq (\alpha+\beta+\alpha k-1)(\alpha+\beta+\alpha k-2) \geq k+1$$

if

$$0 < \alpha \leq 1, \ k \geq k_1,$$

with the natural number k_1 given by (40).

From (41), (42) follows the estimate

$$|\mu(z,m)| \leq \sum_{k=m+1}^{\infty} \left| \frac{(k+1)z^k}{\Gamma(\alpha+\beta+\alpha k)} \right| \leq \sum_{k=m+1}^{\infty} |z|^k = \frac{|z|^{m+1}}{1-|z|}, \ m+1 \geq k_1,$$

and we finally get

$$|\mu(z)| := |\mu(z,k_0)| \leq \rho, \ k_0 = \max\{k_1, \ [\ln(\rho(1-|z|))/\ln(|z|)]\}.$$

■

Now we consider the case B): $|z| > q$.

In this case we use the formula (see Dzherbashyan [11])

$$E'_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) - (\beta-1)E_{\alpha,\beta}(z)}{\alpha z}, \quad (43)$$

which reduces the calculation of the derivative of the Mittag-Leffler function to that of the function $E_{\alpha,\beta}(z)$, already worked out.

5. Numerical Algorithm

The numerical scheme for computation of the Mittag-Leffler function given in pseudocode notation by the algorithm below is based on the results presented in Section 3. The algorithm uses the defining series (2) for arguments z of small magnitude, its asymptotic representations (6), (7) for arguments z of large magnitude, and special integral representations for intermediate values of the argument that include a monotonic part $\int K(\alpha,\beta,\chi,z)\,d\chi$ and an oscillatory part $\int P(\alpha,\beta,\epsilon,\phi,z)\,d\phi$, which can be evaluated using standard techniques.

GIVEN $\alpha > 0$, $\beta \in \mathbf{R}$, $z \in \mathbf{C}$, $\rho > 0$ THEN
IF $1 < \alpha$ THEN
  $k_0 = \lfloor \alpha \rfloor + 1$
  $E_{\alpha,\beta}(z) = \frac{1}{k_0}\sum_{k=0}^{k_0-1} E_{\alpha/k_0,\beta}(z^{\frac{1}{k_0}}\exp(\frac{2\pi i k}{k_0}))$
ELSIF $z = 0$ THEN
  $E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)}$
ELSIF $|z| < 1$ THEN
  $k_0 = \max\{\lceil \frac{(1-\beta)}{\alpha} \rceil,\ \lceil \ln[\rho(1-|z|)]/\ln(|z|) \rceil\}$
  $E_{\alpha,\beta}(z) = \sum_{k=0}^{k_0} \frac{z^k}{\Gamma(\beta+\alpha k)}$
ELSIF $|z| > \lfloor 10 + 5\alpha \rfloor$ THEN
  $k_0 = \lfloor -\ln(\rho)/\ln(|z|) \rfloor$
  IF $|\arg z| < \frac{\alpha\pi}{4} + \frac{1}{2}\min\{\pi,\ \alpha\pi\}$ THEN
    $E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{1/\alpha}} - \sum_{k=1}^{k_0} \frac{z^{-k}}{\Gamma(\beta-\alpha k)}$
  ELSE
    $E_{\alpha,\beta}(z) = -\sum_{k=1}^{k_0} \frac{z^{-k}}{\Gamma(\beta-\alpha k)}$
ELSE
  $\chi_0 = \begin{cases} \max\{1,\ 2|z|,\ (-\ln(\frac{\pi\rho}{6}))^{\alpha}\}, & \beta \geq 0 \\ \max\{(|\beta|+1)^{\alpha},\ 2|z|,\ (-2\ln(\frac{\pi\rho}{[6(|\beta|+2)(2|\beta|)^{|\beta|}]}))^{\alpha}\}, & \beta < 0 \end{cases}$
  $K(\alpha,\beta,\chi,z) = \frac{1}{\alpha\pi}\chi^{\frac{(1-\beta)}{\alpha}} \exp(-\chi^{\frac{1}{\alpha}}) \frac{\chi\sin[\pi(1-\beta)] - z\sin[\pi(1-\beta+\alpha)]}{\chi^2 - 2\chi z\cos(\alpha\pi) + z^2}$
  $P(\alpha,\beta,\epsilon,\phi,z) = \frac{1}{2\alpha\pi}\epsilon^{1+\frac{(1-\beta)}{\alpha}} \exp(\epsilon^{\frac{1}{\alpha}}\cos(\frac{\phi}{\alpha})) \frac{\cos(\omega)+i\sin(\omega)}{\epsilon\exp(i\phi)-z}$
  $\omega = \phi(1 + \frac{(1-\beta)}{\alpha}) + \epsilon^{\frac{1}{\alpha}}\sin(\frac{\phi}{\alpha})$
  IF $|\arg z| > \alpha\pi$ THEN
    IF $\beta \leq 1$ THEN
      $E_{\alpha,\beta}(z) = \int_0^{\chi_0} K(\alpha,\beta,\chi,z)\,d\chi$
    ELSE
      $E_{\alpha,\beta}(z) = \int_1^{\chi_0} K(\alpha,\beta,\chi,z)\,d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,1,\phi,z)\,d\phi$
  ELSIF $|\arg z| < \alpha\pi$ THEN
    IF $\beta \leq 1$ THEN
      $E_{\alpha,\beta}(z) = \int_0^{\chi_0} K(\alpha,\beta,\chi,z)\,d\chi + \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{1/\alpha}}$
    ELSE

$$E_{\alpha,\beta}(z) = \int_{\frac{|z|}{2}}^{\chi_0} K(\alpha,\beta,\chi,z)\,d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,\tfrac{|z|}{2},\phi,z)\,d\phi + \frac{1}{\alpha} z^{\frac{(1-\beta)}{\alpha}} e^{z^{1/\alpha}}$$

ELSE

$$E_{\alpha,\beta}(z) = \int_{\frac{(|z|+1)}{2}}^{\chi_0} K(\alpha,\beta,\chi,z)\,d\chi + \int_{-\alpha\pi}^{\alpha\pi} P(\alpha,\beta,\tfrac{(|z|+1)}{2},\phi,z)\,d\phi$$

END

Remark 5.1. The formulas for $E_{\alpha,\beta}(z)$ in this algorithm are in error at most by ρ. It is advisable to take $\rho = \varepsilon_m$ = machine precision.

6. Figures

In this section, some figures generated by aid of the methods described in the paper are presented. We have produced them by using the programming system MATHEMATICA.

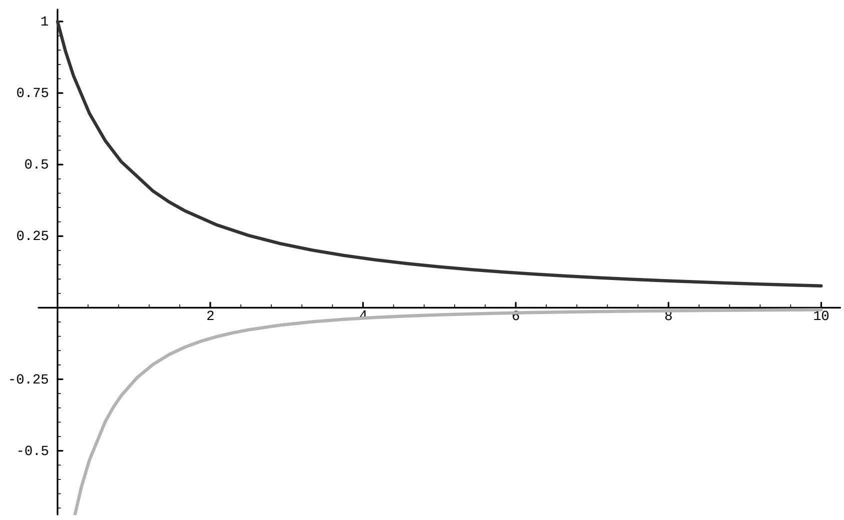

Figure 5: The function $E_{\alpha,\beta}(-t)$ for $\alpha = 0.25,\ \beta = 1$ and its derivative. Black line: $E_{\alpha,\beta}(-t)$, grey line: $\frac{d}{dt}(E_{\alpha,\beta}(-t))$

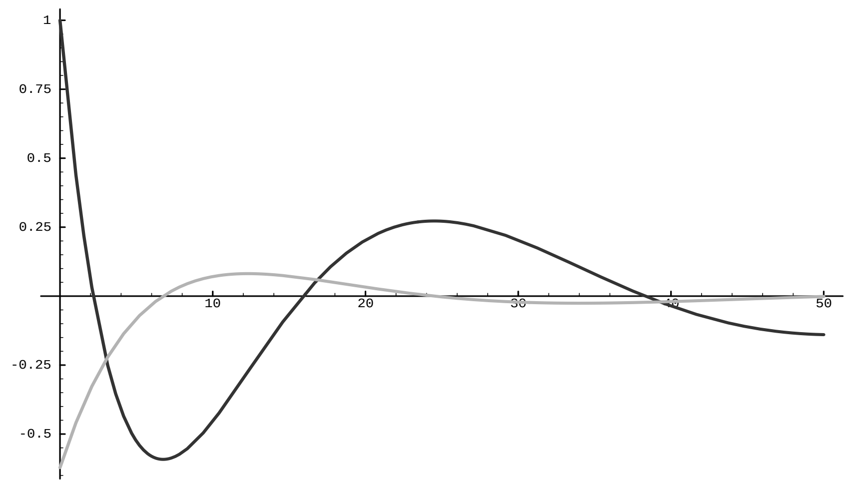

Figure 6: The function $E_{\alpha,\beta}(-t)$ for $\alpha = 1.75,\ \beta = 1$ and its derivative. Black line: $E_{\alpha,\beta}(-t)$, grey line: $\frac{d}{dt}(E_{\alpha,\beta}(-t))$

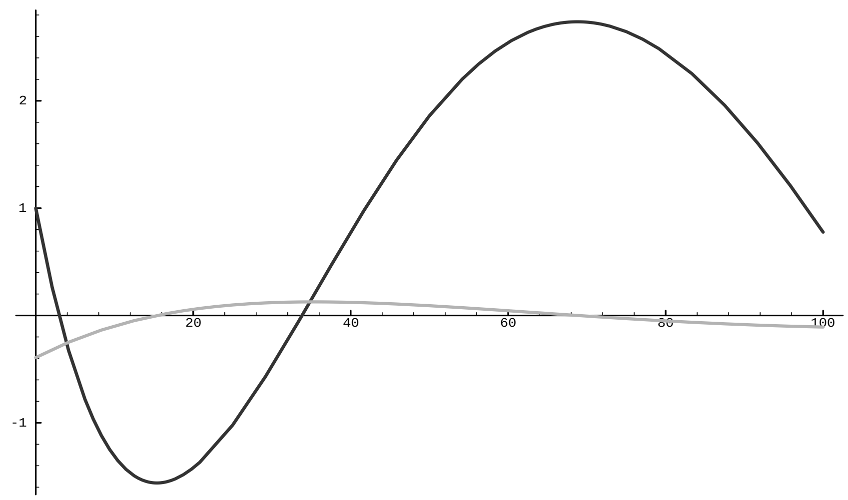

Figure 7: The function $E_{\alpha,\beta}(-t)$ for $\alpha = 2.25,\ \beta = 1$ and its derivative. Black line: $E_{\alpha,\beta}(-t)$, grey line: $\frac{d}{dt}(E_{\alpha,\beta}(-t))$

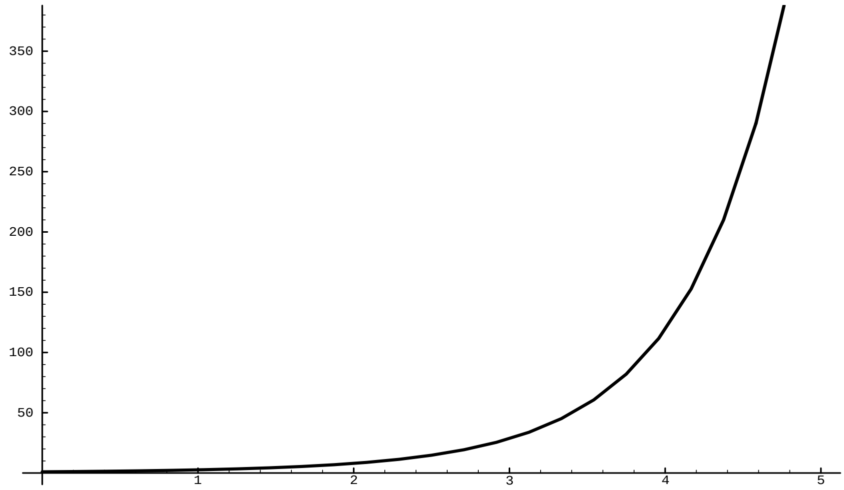

Figure 8: $|E_{\alpha,\beta}(z)| \to \infty$ for $\alpha = 0.75,\ \beta = 1,\ \arg(z) = \frac{\alpha\pi}{4}$

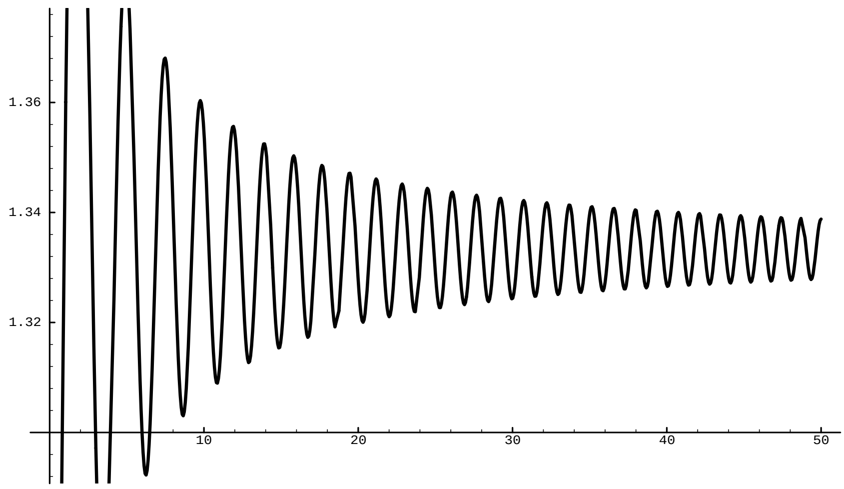

Figure 9: $|E_{\alpha,\beta}(z)| \to \frac{1}{\alpha}$ for $\alpha = 0.75,\ \beta = 1,\ \arg(z) = \frac{\alpha\pi}{2}$

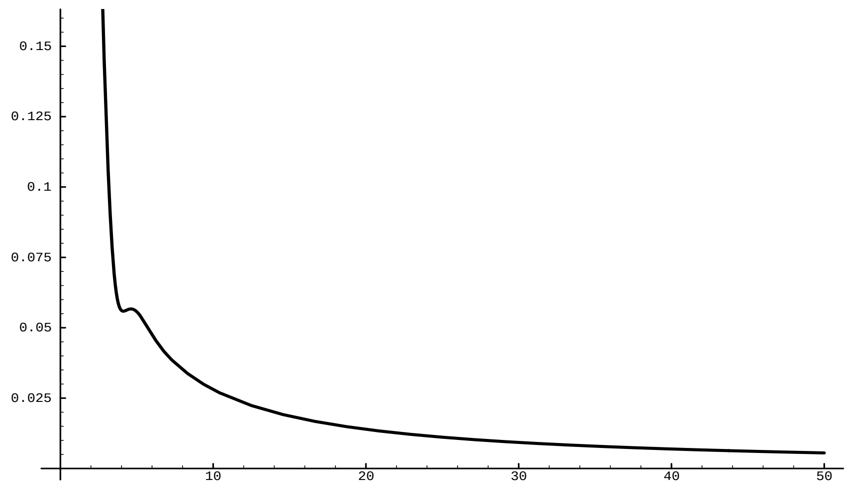

Figure 10: $|E_{\alpha,\beta}(z)| \to 0$ for $\alpha = 0.75,\ \beta = 1,\ \arg(z) = \frac{3\alpha\pi}{4}$

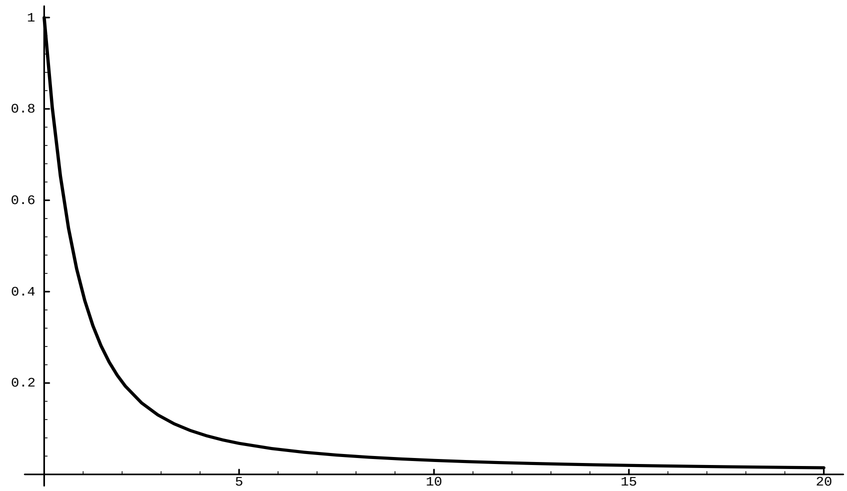

Figure 11: $E_{\alpha,\beta}(z)$ for $\alpha = 0.75,\ \beta = 1,\ \arg(z) = \pi$

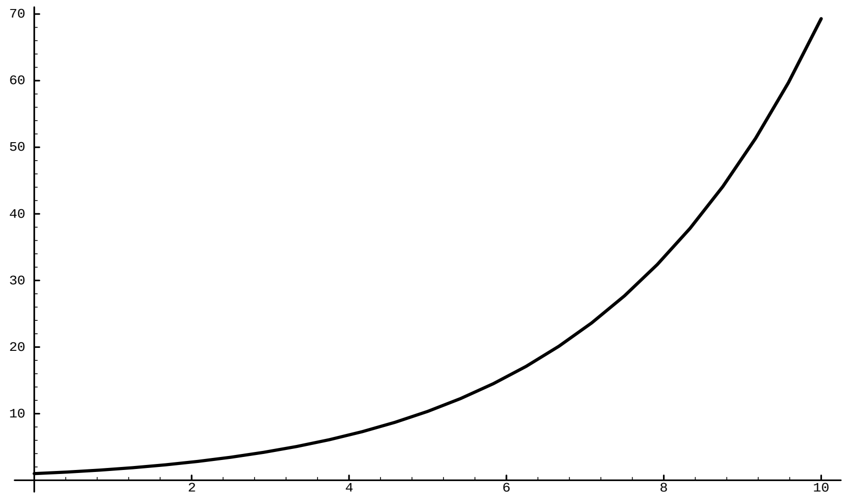

Figure 12: $|E_{\alpha,\beta}(z)| \to \infty$ for $\alpha = 1.25,\ \beta = 1,\ \arg(z) = \frac{\alpha\pi}{4}$

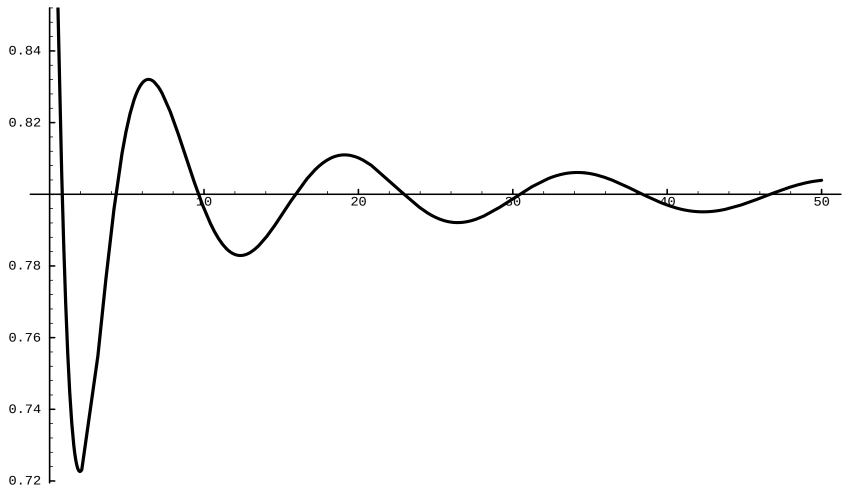

Figure 13: $|E_{\alpha,\beta}(z)| \to \frac{1}{\alpha}$ for $\alpha = 1.25,\ \beta = 1,\ \arg(z) = \frac{\alpha\pi}{2}$

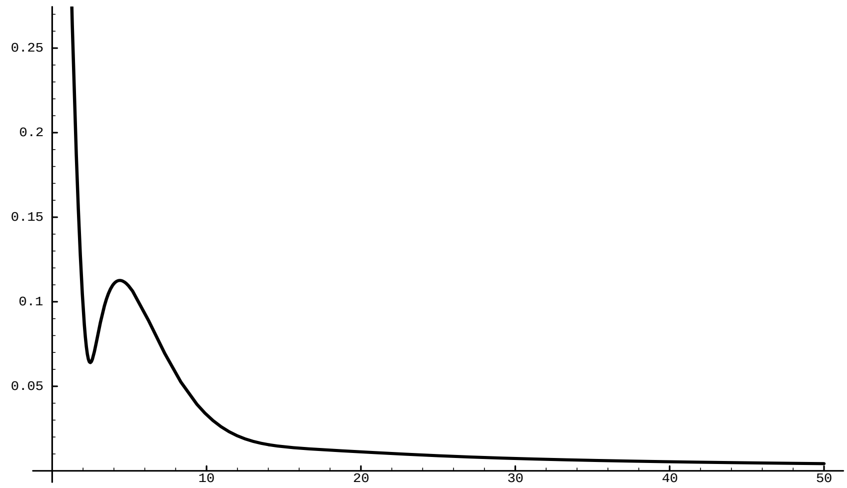

Figure 14: $|E_{\alpha,\beta}(z)| \to 0$ for $\alpha = 1.25,\ \beta = 1,\ \arg(z) = \frac{3\alpha\pi}{4}$

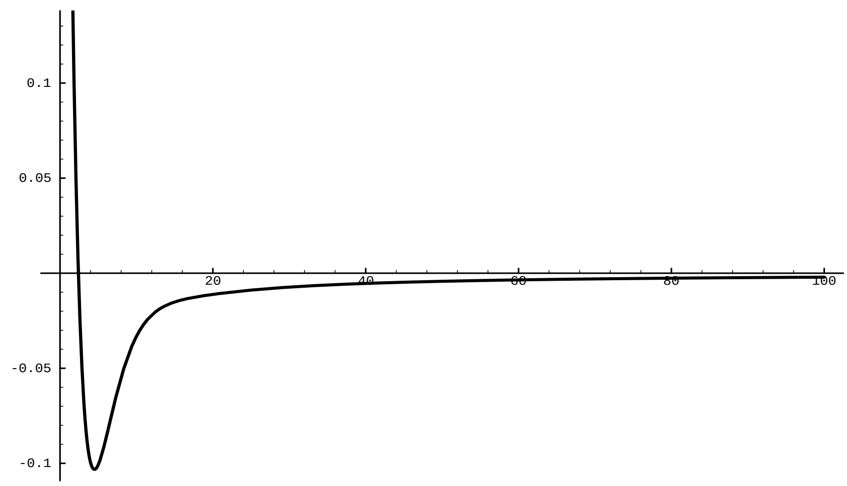

Figure 15: $E_{\alpha,\beta}(z)$ for $\alpha = 1.25,\ \beta = 1,\ \arg(z) = \pi$

References

[1] R.P. A g a r w a l, A propos d'une note de M. Pierre Humbert, *C. R. Acad. Sci. Paris*, **236**(1953), 2031-2032.

[2] Yu.I. B a b e n k o, *Heat and Mass Transfer*, Leningrad, Khimiya, 1986 (in Russian).

[3] R.L. B a g l e y and P.J. T o r v i k, On the appearance of the fractional derivative in the behaviour of real materials, *J. Appl. Mech.* **51**(1984), 294-298.

[4] G.W.S. B l a i r, Psychorheology: links between the past and the present, *Journal of Texture Studies*, **5**(1974), 3-12.

[5] G.W.S. B l a i r, The role of psychophysics in rheology, *J. of Colloid Sciences*, 1947, 21-32.

[6] E. B u c k w a r and Yu. L u c h k o, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations, *Journal of Mathematical Analysis and Applications*, **227**(1998), 81-97

[7] A. B u h l, *Séries analytiques. Sommabilité*, Mem. Sci. Math. Fasc. 7, Gauthier-Villars, Paris.

[8] M. C a p u t o and F. M a i n a r d i, Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento*, Ser II, **1**(1971), 161-198.

[9] C l e o t a and H u g h e s, Asymptotic developments of certain integral functions, *Duke Math. J.*, **9**(1942), 791-802.

[10] M.M. D z h e r b a s h y a n, About integral representation of functions continuous on some rays (generalization of Fourier integral), *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.*, **18**(1954), 427-448. (In Russian).

[11] M.M. D z h e r b a s h y a n, *Integral Transforms and Representations of Functions in the Complex Plane*, Nauka, Moscow, 1966 (in Russian).

[12] A.M.A. E l - S a y e d, Fractional order diffusion-wave equation, *J. of Theoretical Physics* **35**, No 2 (1996), 371-382.

[13] A. E r d é l y i, W. M a g n u s, F. O b e r h e t t i n g e r and F.G. T r i c o m i, *Higher Transcendental Functions*, Vol. 3, New York, McGraw-Hill Book Co., 1954.

[14] R. G o r e n f l o, Yu. L u c h k o and F. M a i n a r d i, Analytical properties and applications of the Wright function, *Fractional Calculus and Applied Analysis* **2**(1999), 383-414.

[15] R. G o r e n f l o, Yu. L u c h k o and F. M a i n a r d i, Wright functions as scale-invariant solutions of the diffusion-wave equation, *Journal of Computational and Applied Mathematics* **118**(2000), 175-191.

[16] R. G o r e n f l o, Yu. L u c h k o and P.P. Z a b r e j k o, On solvability of linear fractional differential equations in Banach spaces, *Fractional Calculus and Applied Analysis* **2**(1999), 163-176.

[17] R. G o r e n f l o and F. M a i n a r d i, Fractional oscillations and Mittag-Leffler functions, Preprint No. A-14/96, Free University of Berlin, Berlin, 1994.

[18] R. G o r e n f l o and F. M a i n a r d i, The Mittag-Leffler function in the Riemann-Liouville fractional calculus, In: Kilbas A.A. (eds.) *Boundary value problems, special functions and fractional calculus*, Minsk, 1996, 215-225.

[19] R. G o r e n f l o and F. M a i n a r d i, Fractional Calculus: Introduction to Integral and Differential Equations of Fractional Order, in: *Fractals and Fractional Calculus in Continuum Mechanics* (Ed. A. Carpinteri and F. Mainardi), Springer Verlag, Wien, 1997.

[20] R. G o r e n f l o and R. R u t m a n, On ultraslow and intermediate processes, In: Rusev P., Dimovski I. and Kiryakova V. (eds.) *Transform Methods and Special Functions*, Sofia, 1994, 61-81.

[21] R. G o r e n f l o and S. V e s s e l l a, *Abel Integral Equations: Analysis and Applications*, Lecture Notes in Mathematics **1461**, Springer-Verlag, Berlin, 1991.

[22] P. H u m b e r t, Quelques resultats relatifs a la fonction de Mittag-Leffler, *C. r. Acad. sci. Paris*, **236**(1953), 1467-1468.

[23] P. H u m b e r t and R.P. A g a r w a l, Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations, *Bull. sci. math. (2)*, **77**(1953), 180-185.

[24] Yu. L u c h k o, Operational method in fractional calculus, *Fractional Calculus and Applied Analysis* **2**(1999), 463-488.

[25] Yu. L u c h k o, Asymptotics of zeros of the Wright function, *Journal for Analysis and its Applications* **19**(2000), 584-595.

[26] Yu. L u c h k o and R. G o r e n f l o, Scale-invariant solutions of a partial differential equation of fractional order, *Fractional Calculus and Applied Analysis*, **1**(1998), 63-78.

[27] Yu. L u c h k o and H.M. S r i v a s t a v a, The exact solution of certain differential equations of fractional order by using operational calculus, *Computers Math. Appl.* **29**(1995), 73-85.

[28] F. M a i n a r d i, Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics, in: *Fractals and Fractional Calculus in Continuum Mechanics* (Ed. A. Carpinteri and F. Mainardi), Springer Verlag, Wien, 1997.

[29] F. M a i n a r d i, Yu. L u c h k o and G. P a g n i n i, The fundamental solution of the space-time fractional diffusion equation, *Fractional Calculus and Applied Analysis* **4**(2001), 153-192.

[30] W.R. S c h n e i d e r and W. W y s s, Fractional diffusion and wave equations, *J. Math. Phys.* **30**(1989), 134-144.

[31] G.L. S l o n i m s k i, About a law of deformation of polymers, *Dokl. Akad. Nauk SSSR (Physic)*, **140**(1961), No 2, 343-346.

[32] S. W e s t e r l u n d and L. E k s t a m, Capacitor theory, *IEEE Trans. on Dielectrics and Electrical Insulation* **1**(1994), 826-839.

[33] A. W i m a n, Über den Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$, *Acta Math.*, **29**(1905), 191-201.

[34] A. W i m a n, Über die Nullstellen der Funktionen $E_\alpha(x)$, *Acta Math.*, **29**(1905), 217-234.

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