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FRACTIONAL CALCULUS :
Some Numerical Methods

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ABSTRACT

A survey is given on some numerical methods of Riemann-Liouville fractional calculus. The topics discussed here will be: (a) approximation of fractional derivatives by generalized finite differences and their use in numerical treatment of fractional differential equations, (b) discretized fractional calculus and its use in numerical treatment of Abel type integral equations of first and second kind, (c) product integration and collocation methods for Abel integral equations, (d) the problem of ill-posedness of Abel integral equations of first kind.

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1. INTRODUCTION

We shall describe and discuss a few methods of numerical treatment of ordinary fractional differential equations and Abel integral equations of first and second kind.

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There exists a rich literature on these subjects, Abel integral equations being a special sort of Volterra integral equations, and much material and many references on theory and practice can be found in the books by Brunner and van der Houwen [1] and by Linz [2]. For the analytical theory of the relevant fractional integral and differential operators we recommend the books by Samko, Kilbas and Marichev [3] and by K.S. Miller and B. Ross [4]. Let us also mention the book by Gorenflo and Vessella [5]. Concerning the basic theory of fractional differential equations and their applications and references we refer the reader to the papers by Gorenflo and Mainardi [6] and by Mainardi [7] that he will find in this volume.

The linear operators occurring in the sequel and to be approximated are the *Riemann-Liouville operator* J^α *of fractional integration* and the *operator* D^α *of fractional differentiation*, α being nonnegative.

Here we do not hesitate to repeat the definitions introduced in [6] in order to make this contribution self-consistent. The operator J^α is defined by the formula

$$J^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, u(\tau)\, d\tau, \quad t>0\,, \quad \alpha>0\,, \tag{1.1}$$

whereas its left-inverse D^α is given as

$$D^\alpha := D^m\, J^{m-\alpha}\,, \quad m-1<\alpha\le m\,, \quad m\in \mathbb{N}\,, \tag{1.2}$$

where D^m denotes the ordinary derivative of order m . For complementation we define

$$J^0 = D^0 = I \quad \text{(Identity operator)}. \tag{1.3}$$

In particular, we have

$$Ju(t) = \int_0^t u(\tau)\, d\tau, \quad t>0\,.$$

These formulas, of course, require that the function u is sufficiently well behaved. We note the *semigroup property*

$$J^\alpha J^\beta = J^{\alpha+\beta} \quad \text{for} \quad \alpha\ge 0,\ \beta\ge 0, \tag{1.4}$$

and the formula (for $\alpha \ge 0$)

$$D^\alpha J^\alpha = I,$$

which means

$$D^\alpha J^\alpha u(t) = u(t). \tag{1.5}$$

Remark: In order to remain in accordance with the standard notation I for the Identity operator we use the character J for the integral operator and its power J^α.

For convenience, we define the operator $J^\alpha_{-\infty}$ by

$$J^\alpha_{-\infty} u(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1}\, u(\tau)\, d\tau\,, \quad t \in \mathbb{R}\,, \quad \alpha > 0\,. \tag{1.6}$$

If $u(t)$ is a *causal* function, *i.e.* $u(t) = 0$ for $-\infty < t < 0$ then

$$J^\alpha_{-\infty} u(t) = J^\alpha u(t) \quad \text{for} \quad t > 0. \tag{1.7}$$

By $J^\alpha u(0)$ we mean the limit (if it exists) of $J^\alpha u(t)$ for $t \to 0^+$. This limit may be infinite. It is useful to have in mind the effect of our operators J^α and D^α on the power functions t^γ with $\gamma > -1,\ t > 0$. We have, for $\alpha \ge 0$, the relations

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \quad D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,. \tag{1.8}$$

For proofs consult [3], [4] or [8].

Note the remarkable fact that the fractional derivative $D^\alpha u$ is not zero for the constant function $u(t) \equiv 1$ if $\alpha \notin \mathbb{N}$. In fact, (1.8) with $\gamma = 0$ teaches us that

$$D^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad \alpha \ge 0, \quad t > 0. \tag{1.9}$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function in the points $0, -1, -2, \ldots$. Furthermore, we observe, again by looking at (1.8), that

$$D^\alpha t^{\alpha-1} \equiv 0 \quad \text{for} \quad t > 0, \quad \alpha > 0,$$

which implies that D^α is not right-inverse to J^α. We have

$$J^\alpha D^\alpha t^{\alpha-1} \equiv 0, \quad \text{but} \quad D^\alpha J^\alpha t^{\alpha-1} = t^{\alpha-1} \quad \text{for} \quad t > 0, \quad \alpha > 0.$$

These matters cause some problems in numerical treatment of fractional differential and integral equations and require great care in analytical investigations.

Everything would be more coherent if we would consistently work with the operators $J^\alpha_{-\infty}$ and $D^\alpha_{-\infty} = D^m J^{m-\alpha}_{-\infty}$, $m-1 < \alpha \le m$, $m \in \mathbb{N}$, and with generalized functions in the sense of Gel'fand and Shilov [9] instead of functions, these generalized functions vanishing for $t < 0$. In the case of continuous functions $u(t)$ having continuous derivatives up to order m for $t \ge 0$, an alternative would be to use the Caputo fractional derivative $D^\alpha_* = J^{m-\alpha} D^m$ as we have described in [6]. However, in order to remain in the mainstream of published work in fractional calculus, we use classical functions defined for $t \ge 0$, and the operators J^α and D^α.

For numerical treatment of fractional differential and integral equations it is essential to have available good approximations of the operators D^α and J^α of fractional differentiation and integration. So, in Section 2 we shall discuss the approach of Grünwald [10] and Letnikov [11] for approximation of D^α by generalized finite differences, and in Section 3 we shall describe the basic ideas of discretized fractional calculus as it has been developed by Lubich [12]. In Section 4 we shall give a short survey on literature on product integration and collocation methods for Abel type integral equations, and on the problem of ill-posedness of Abel integral equations of first kind. This ill-posedness causes specific troubles in many situations where a situation of evaluation of physical measurements is modeled by use of an Abel integral equation of first kind (see [5], [8] and [13] for examples).

Throughout, our presentation will be informal, all occurring functions being assumed to be so regular that what is written down is meaningful. For precise conditions of required smoothness etc. and for rigorous proofs the reader is advised to look into the quoted references.

2. GENERALIZED FINITE DIFFERENCES

For derivation of approximations to the fractional differentiation operator D^α it is convenient to make use of the *discrete operators of translation (shift) and finite differences.* In a lucid way the theory of numerical differentiation and integration (with equidistant grid points) is developed in Chapters 7 to 10 of [14]. See also Chapter 6 of [15].

Let be $\tau \in \mathbb{R}$. Then we define the shifting operator E^τ and the (backward, forward, central) difference operators ∇_τ, Δ_τ and δ_τ by their action on a function $u(t)$ defined for $t \in \mathbb{R}$.

$$\begin{cases} E^\tau u(t) = u(t+\tau)\,, \\ \nabla_\tau u(t) = u(t) - u(t-\tau)\,, \\ \Delta_\tau u(t) = u(t+\tau) - u(t)\,, \\ \delta_\tau u(t) = u(t+\tau/2) - u(t-\tau/2)\,. \end{cases} \tag{2.1}$$

Obviously, the operators E^τ for $\tau \in \mathbb{R}$ have the group property,

$$E^{\sigma+\tau} = E^\sigma E^\tau, \quad \sigma \in \mathbb{R}, \quad \tau \in \mathbb{R}\,, \tag{2.2}$$

meaning $E^\sigma E^\tau u(t) = E^\sigma u(t+\tau) = u(t+\tau+\sigma)$, and it is for this reason that in E^τ we use τ as an exponent instead of as an index. We furthermore have the relations, with I as Identity operator,

$$\nabla_\tau = I - E^{-\tau}, \quad \Delta_\tau = E^\tau - I, \quad \delta_\tau = \frac{1}{2}\left(E^{\tau/2} - E^{-\tau/2}\right)\,, \tag{2.3}$$

and all operators of this group commute with each other.

With these notations, we can write the well-known approximations $[u(t) - u(t - h)]/h$ and $[u(t+h/2) - u(t-h/2)]/h$ for the derivative $u'(t)$ of a differentiable function $u(t)$ as $[\nabla_h u(t)]/h$ and $[\delta_h u(t)]/h$, with $h > 0$ small. For $h \to 0$ these approximations have $O(h)$ and $O(h^2)$, respectively, as order of accuracy, if the function $u(t)$ is sufficiently smooth. See again [14] or [15].

Higher order derivatives $u^{(n)}(t) = D^n u(t)$ with $n \in \mathbb{N}$ can, with $h > 0$ small, be approximated by

$$[\nabla_h^n u(t)]/h^n = h^{-n}(I - E^{-h})^n u(t)$$

or

$$[\delta_h^n u(t)]/h^n = h^{-n} \left(E^{h/2} - E^{-h/2}\right)^n u(t)$$

again in case of $u(t)$ being sufficiently smooth, with order of accuracy $O(h)$ or $O(h^2)$, respectively. The powers ∇_h^n and δ_h^n can readily be expanded via the binomial theorem:

$$\nabla_h^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} E^{-jh},$$

$$\delta_h^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} E^{(n-j)\,h/2} E^{-j\,h/2} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} E^{(n/2-j)\,h}.$$

This leads to the well known formulas

$$h^{-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} u(t - jh) = D^n u(t) + O(h), \tag{2.4}$$

$$h^{-n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} u\left(t + (n/2 - j)\,h\right) = D^n u(t) + O(h^2). \tag{2.5}$$

In passing, let us remark that also

$$h^{-n} \Delta^n u(t) = D^n u(t) + O(h).$$

However, for applications to causal problems backward operators are more appropriate.

The remarkable fact now is that these formulas can be generalized to the case of noninteger order of derivative. Replacing the positive integer n by a positive real number α amounts to use the formal powers

$$\nabla_h^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} E^{-j\,h}, \quad \delta_h^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} E^{(\alpha/2-j)\,h},$$

in analogy to the expansions (E^{-h} replaced by the complex variable z)

$$(1-z)^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} z^j,$$

$$(z^{-1/2} - z^{1/2})^\alpha = z^{-\alpha/2} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} z^j = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} z^{j-\alpha/2},$$

which are convergent if $|z| < 1$. We thus obtain the *Grünwald-Letnikov approximation*

$$h^{-\alpha}\, \nabla_h^\alpha u(t) = h^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} u(t-jh) = D^\alpha u(t) + O(h) \tag{2.6}$$

and the difference approximation

$$h^{-\alpha}\, \delta_h^\alpha u(t) = h^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} u\left(t + (\alpha/2 - j)\, h\right) = D^\alpha\, u(t) + O(h^2). \tag{2.7}$$

These formulas reduce to (2.4) and (2.5) if $\alpha = n \in \mathbb{N}$.

In [3] precise sufficient conditions are given for convergence of $\nabla_h^\alpha u(t)$ towards $u(t)$ as $h \to 0$. A necessary condition, of course, is that the infinite series does converge which certainly is the case if $u(t)$ decays towards zero sufficiently fast as $t \to -\infty$, in particular if $u(t) = 0$ for all $t < 0$. This can naturally be supposed if $u(t)$ is solution of an evolutionary fractional differential equation with starting point $t = 0$. Hence, if $u(t) = 0$ for $t < 0$,

$$h^{-\alpha}\, \nabla_h^\alpha u(t) = h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^j \binom{\alpha}{j} u(t-jh), \tag{2.8}$$

$$h^{-\alpha}\, \delta_h^\alpha u(t) = h^{-\alpha} \sum_{j=0}^{[t/h+\alpha/2]} (-1)^j \binom{\alpha}{j} u(t + (\alpha/2 - j)\, h)\,. \tag{2.9}$$

However, these formulas have order $O(h)$ or $O(h^2)$, respectively, of accuracy only if transition of $u(t)$ to zero for $t < 0$ is sufficiently smooth at the origin $t = 0$. From [16] one can take, among other things, that existence of integrable derivatives $D^n u(t)$ up to $n = [\alpha] + 3$ or $n = [\alpha] + 4$, respectively, is sufficient for uniform $O(h)$ or $O(h^2)$ accuracy in an arbitrary bounded interval $[0, T]$.

If one wants to numerically differentiate a function $u(t)$ given as a smooth function for $t \geq 0$ then the transition to its extension by zero for negative argument t may be non-smooth at the origin $t = 0$, even discontinuous, and the approximations (2.8) and (2.9) are of dubious value and may cause trouble. However, often this singularity, induced by the zero extension, has the form of a generalized polynomial

$$p(t) = c_0 t^{\beta_0} + c_1 t^{\beta_1} + \ldots + c_k t^{\beta_k}, \quad t \geq 0\,, \qquad p(t) = 0\,, \quad t < 0\,,$$

which can be subtracted to obtain a function sufficiently smooth in all of $\mathbb{R}$ and vanishing for negative argument. By (1.8) now $p(t)$ can be fractionally differentiated exactly,

$$D^\alpha p(t) = \sum_{j=0}^{k} \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 1 - \alpha)}\, t^{\beta_j - \alpha}, \quad \text{if all} \quad \beta_j > -1\,.$$

If the coefficients c_j are not known but $-1 < \beta_0 < \beta_1 < \beta_2 < \ldots$, then good knowledge of the function u can be used to calculate or estimate them. For example, $c_0 = \lim\,[t^{-\beta_0} u(t)]$ as $t \to 0^+$.

When trying to approximate the solution $u(t)$ of an ordinary fractional differential equation, given for $t \geq 0$, by a grid function $u_h(jh)$ (with a "small" positive steplength h), $j = 0, 1, 2, \ldots$, one can make use of (2.8) by replacing all occurrences of fractional derivatives $D^\alpha u(t)$ by

$$h^{-\alpha}\, \nabla_h^\alpha u_h(kh) = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j}\, u_h((k-j)h), \quad k = 1, 2, 3, \ldots$$

and recursively solving for the new value $u_h(kh)$. Troubles arise if at the origin $u(t)$ is not smoothly extendable to zero for negative argument, and further research how to overcome these is required.

Another problem is the problem of stiffness. Depending on the sign of the coefficients of the various terms in a linear differential equation (fractional or non-fractional) it may be necessary to restrict the size of h in order to have at first solvability at each step of increasing the index k and at second numerical stability, and there are instances where h must be chosen extremely small.

Podlubny [17] has applied this method successfully for solving with $O(h)$ accuracy (a) the fractional relaxation-oscillation equation, (b) the Bagley-Torvik equation of an immersed plate, namely

(a) $D^\alpha y(t) + Ay(t) = f(t), \quad t > 0\,, \quad A > 0\,, \quad 0 < \alpha < 2\,,$
with $y(0^+) = 0$ in case $0 < \alpha \leq 1\,,$ $y(0^+) = y'(0^+) = 0$ in case $1 < \alpha < 2\,;$

(b) $Ay''(t) + B\, D^{3/2} y(t) + Cy(t) = f(t), \quad t > 0\,, \quad A > 0\,,$ with $y(0^+) = y'(0^+) = 0\,.$

In these examples the special choice of initial conditions ensures sufficient smoothness of transition at the origin $t = 0$ to the zero extension for negative argument.

3. DISCRETIZED FRACTIONAL CALCULUS

For numerical treatment of fractional integral equations, in particular linear and nonlinear Abel-type integral equations of first and second kind, but also of integro-differential equations with fractional integral operators, it is important to have good methods for approximating expressions

$$J^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \quad t > 0, \quad \alpha > 0. \tag{3.1}$$

We give a short description of Lubich's basic ideas of discretized fractional calculus. See his and his co-authors' and colleagues' papers [12] and [18-22] for description of the theory and applications to linear and non-linear Abel type integral equations and ordinary integro-differential equations. For applications to partial integro-differential equations containing fractional integration operators we recommend to look into the papers [23-26] by Sanz-Serna, López-Marcos, Xu-Da and Lubich.

The idea is to approximate (3.1) by a discrete *convolution quadrature*:

$$J_h^{\alpha} u(t) = h^{\alpha} \sum_{j=0}^{n} \omega_{n-j} u(jh) + h^{\alpha} \sum_{j=0}^{s} w_{nj} u(jh)\,, \tag{3.2}$$

where $t = nh,\ h > 0,\ n \in \mathbb{N},\ s \in \mathbb{N}_0$, with appropriate convolution quadrature weights $\omega_k,\ k \geq 0$, and starting quadrature weights w_{nj}, $n \geq 0,\ j = 0, 1, \ldots, s$, that do not depend on h. The index s of summation is fixed. These methods can be adjusted to yield higher orders of accuracy in contrast to the order 1 of the simple Grünwald-Letnikov approach. However, this higher order must be paid by a lot of work in calculating the coefficients ω_k and w_{nj}. For efficiency of these calculations (for large values of n) Lubich and co-workers recommend to use the fast Fourier transform.

The starting weights w_{nj} are required to cope with the possibly singular behaviour of the function u at the origin $t = 0$, and the character of this behaviour is crucially used in calculating them. We here do not go into the details but refer the reader to [12]. However, we describe the idea for finding the convolution quadrature weights ω_k.

The mathematical tool is the manipulation of *power series of generating functions*, in generalizing analogy to their use in difference schemes for ordinary differential equations (see [27]). We consider a linear multistep method for approximating

$$y(t) = Ju(t) = \int_0^t u(\tau)\, d\tau$$

by values $y_n \approx y(t_n)$, with $t_n = nh,\ n = 0, 1, 2, \ldots$. Put $u_n = u_h(nh)$ for $n \geq 0$, $u_n = 0$ for $n < 0$, and let z symbolize the discrete backward shift operator:

$$zu_n = u_{n-1}\,, \quad z^k u_n = u_{n-k}.$$

A general linear multistep method has, with $\alpha_k \neq 0$, the form

$$\alpha_k \tilde{y}_n + \alpha_{k-1}\tilde{y}_{n-1} + \ldots + \alpha_0 \tilde{y}_{n-k} = h\{\beta_k u_n + \beta_{k-1}u_{n-1} + \ldots + \beta_0 u_{n-k}\}$$

with given coefficients $\alpha_j,\ \beta_j$. With the formal polynomials

$$\tilde{\rho}(z) = \alpha_k + \alpha_{k-1}z + \ldots + \alpha_0 z^k\,, \quad \tilde{\sigma}(z) = \beta_k + \beta_{k-1}z + \ldots + \beta_0 z^k\,,$$

and the formal rational function

$$\omega(z) = \frac{\tilde{\sigma}(z)}{\tilde{\rho}(z)}$$

symbolically

$$\tilde{\rho}(z)\tilde{y}_n = h\tilde{\sigma}(z)u_n,$$

$$\tilde{y}_n = h\omega(z)u_n.$$

In general,

$$\omega(z) = \omega_0 + \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \ldots$$

comes out as a power series with infinitely many coefficients ω_j different from zero. And then

$$\tilde{y}_n = h\sum_{j=0}^{\infty} \omega_j u_{n-j}.$$

But only finitely many u_k being nonvanishing, this series actually terminates with upper summation index $j = n$. However, correction is required because u generally does not have a smooth transition to its zero extension for negative argument. Hence we take

$$y_n = h\omega(z)u_n + \text{ correction terms.}$$

The idea for approximation of $J^\alpha u(t)$ for arbitrary positive α is now to replace the power series $\omega(z)$ by $(\omega(z))^\alpha$, thus taking as approximations for $J^\alpha u(nh)$ the values

$$y_n = h^\alpha(\omega(z))^\alpha u_n + \text{ correction terms.}$$

Lubich describes how appropriate correction terms can be determined in dependence on the smoothness or non-smoothness of transition of the function $u(t)$ at the origin to its zero extension for negative argument t.

He also develops the theory of *consistency*, *stability* and *convergence* in analogy to that of linear multistep quadrature. In particular, he shows that the order of accuracy of the relevant multistep method is also valid for the modification to the fractional case. For this theory, the symbol z of the backward shift operator is treated as a complex variable. We remark that the occurring power series have positive radius of convergence.

Simple Examples

(a) *The backward Euler method*

From the approximation

$$\tilde{y}_n = \tilde{y}_{n-1} + hu_n \approx Ju(nh)$$

we take

$$(1-z)\tilde{y}_n = hu_n\,,$$

hence

$$\tilde{\rho}(z) = 1 - z, \quad \tilde{\sigma}(z) \equiv 1\,,$$

and

$$\omega(z) = \frac{1}{1-z} = (1-z)^{-1} = \sum_{j=0}^{\infty} z^j.$$

Consequently for approximation of $J^\alpha u(nh)$ we use

$$\tilde{y}_n = h^\alpha (1-z)^{-\alpha} u_n = h^\alpha \sum_{j=0}^{\infty} (-1)^j \binom{-\alpha}{j} z^j u_n = h^\alpha \sum_{j=0}^{[t/h]} (-1)^j \binom{-\alpha}{j} u_{n-j}.$$

Observe the analogy to the Grünwald-Letnikov approximation for the fractional derivative (α replaced by $-\alpha$).

(b) *The trapezoidal rule*

From the approximation

$$\tilde{y}_n - \tilde{y}_{n-1} = \frac{h}{2}(u_n + u_{n-1}) \approx Ju(nh)$$

we take

$$(1-z)y_n = \frac{h}{2}(1+z)u_n\,,$$

hence

$$\tilde{\rho}(z) = 1 - z\,, \quad \tilde{\sigma}(z) = \frac{1}{2}(1+z) \quad \omega(z) = \frac{1+z}{2(1-z)}\,,$$

and for approximation of $J^\alpha u(nh)$ we use

$$\tilde{y}_n = h^\alpha \left(\frac{1+z}{2(1-z)}\right)^\alpha u_n\,.$$

4. OTHER METHODS AND EVALUATION OF MEASUREMENTS

As indicated in Section 1 there exists a vast literature on numerical methods for Abel type integral equations, and we resist here the temptation to give a comprehensive overview on these. Let us just mention *product quadrature methods* and *piecewise polynomial collocation methods* on which one may find presentations and references in [1] and [2]. Let us also mention the recent papers [28] by A.P. Orsi on *product integration* and L. Blank [29] on a *spline collocation method* (for fractional differential equations). Short surveys can also be found in [5] and [30].

Completely distinct from the methods described in this survey are methods that make use of integral representations of functions of Mittag-Leffler type. Such representations are very useful in analytically solving fractional differential or integral equations with constant coefficients, and they are often obtainable via clever bending of the path of integration in the Laplace transform inversion formula and using the residue theorem. In particularly nice cases one then has the Laplace transform of a non-negative (or non-positive) function superimposed by sinusoidal oscillatory parts with exponentially decaying amplitudes. The latter can be calculated by standard subroutines, and the Laplace transform integral also does not present serious difficulties (due to the fast decrease of the integrand). The graphs of solutions in [31] have been produced in this way.

Yet another method, applicable to calculate Mittag-Leffler functions of rational index with small denominators, consists of using their representations in terms of the incomplete gamma function (formulas can be found e.g. in [4]) for which also subroutines are available.

In [5], [8] and [30] among other things the serious problem of ill-posedness of Abel integral equation of first kind is discussed. When in such an equation the data function is given by physical measurements, the noise in the measurement process is often amplified to an unacceptable degree by a straightforward numerical solution method, and methods of high order accuracy are worthless. In [5], [8] and [30] a method of *descriptive regularization* is recommended and demonstrated to work well. The trick is to combine a Gauss least squares fitting technique with taking account of extra qualitative properties (like nonnegativity, monotonicity, or convexity) the solution is known to possess.

The problem of using product integration methods for Abel integral equations with special attention to possible nonsmoothness or even discontinuous solutions is extensively treated and illustrated in Ch. Kutsche's thesis [32].

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