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CONSIDERATIONS ON FRACTIONAL CALCULUS: INTERPRETATIONS AND APPLICATIONS
(Contribution to a Round Table Discussion)

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I regret to recognize that, in spite of several and outstanding mathematicians, who since Euler's times have given contributions in the so-called fractional calculus (see *e.g.* [1]-[4] for historical surveys), this topic is still nowadays a "cinderella" in mathematical analysis. Possibly, this is due to the fact that fractional calculus can be introduced in different ways, which are not all equivalent.

In physics and engineering this topic is seen with some suspect because people are accustomed to the physical and geometric interpretation of the ordinary integrals and derivatives, so that they are looking for possible "similar" interpretations of the fractional integrals and derivatives.

I would like to say that, since from Abel's times, the successful applications of the *fractional integrals and derivatives* of order $1/2$ (in the sense of Riemann-Liouville) in some basic problems of mechanics provide by themselves an interpretation, for me satisfactory. In this respect I would like to outline the classical (but mostly ignored) treatise by Davis [5] and the most recent monography by Gorenflo and Vessella [6] and, to a certain extent, the general lectures given at this Workshop by Gorenflo [7] and myself [8]. In my lecture I also mentioned the *application of fractional integrals and derivatives of arbitrary order* α (usually $0 < \alpha < 1$) *in linear viscoelasticity* (more generally, *in hereditary mechanics*). In this field, these objects started to appear long time ago, both explicitly (see *e.g.* [9]-[13]) and implicitly (see *e.g.* [14]-[16]), and in recent times they still continue to appear (see *e.g.* [17]-[21]).

Some problems are of course repetitive or not completely new, surely due to the fact that the theory and applications of fractional calculus are not so popular and thus need a continuous re-discovery. In my opinion, *it would be desirable that elementary textbooks of "calculus" could introduce fractional calculus* (possibly only in the Riemann-Liouville sense) *with some basic applications*. Of course, *also the topic of special functions* could take benefit of this introduction, since some of them, as Gamma, Mittag-Leffler and Wright functions, are closely connected with fractional calculus. More general functions, as Fox H- and Meijer G-functions, would be left to specialists in advanced courses, see *e.g.* [22]-[23]. *Fractional calculus can be considered as a "laboratory" for special functions and integral transforms.* As an example, one can consider the special functions of mathematical physics as "generalized fractional integrals or derivatives" of three basic elementary functions [23]. Also, several generalized Laplace-type integral transforms can be seen as "transmutations" (which are fractional integrals) of the Laplace transform, see [22, Ch.5].

I would like to conclude my intervention with a number of applications of Riemann-Liouville fractional calculus in fields not considered in my lecture [8]. Surely, a promising topic concerns *evolution equations of fractional order that can simulate phenomena intermediate between diffusion and wave propagation and/or processes of anomalous diffusion in complex or fractal systems.* In this respect, we quote the papers [24]-[30] for mathematical aspects, and [31]-[43] for physical applications, without pretending to be exhaustive.

The possible direct connection between fractional calculus and fractals is still a questionable matter and remains nowadays an open problem; however, in view of the large popularity of fractals, this topic can keep alert the community of scientists and play a role in letting know the wonderful world of fractional calculus [44]. On the other hand, *fractals and fractional calculus seem to have in common, not only the first five letters, but also some final (i.e. asymptotic) results related to the power laws.* We hope to be able to tell more in a near future.

References

1. K. B. Oldham & J. Spanier, *The Fractional Calculus*, Academic Press, New York (1974).
2. B. Ross (Editor), *Fractional Calculus and its Applications*, Lecture Notes in Mathematics # 457, Springer Verlag, Berlin (1975). [Proc. Int. Conf. held at Univ. of New Haven (USA), 1974]
3. S. G. Samko, A. A. Kilbas & O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam (1993). [Engl. Transl. from Russian, Nauka i Tekhnika, Minsk (1987)]
4. K. S. Miller & B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).
5. H.T. Davis, *The Theory of Linear Operators*, Principia, Bloomington (1936).
6. R. Gorenflo & S. Vessella, *Abel Integral Equations: Analysis and Applications*, Lect. Notes in Math. # 1461, Springer-Verlag, Berlin (1991).
7. R. Gorenflo, The tomato salad problem in spherical stereology, In: *same volume "TM & SF, Varna'96"*, 132-149.
8. F. Mainardi, Applications of fractional calculus in mechanics, In: *same volume "TM & SF, Varna'96"*, 309-334.
9. A. Gemant, *Frictional Phenomena*, Chemical Publ. Co, Brooklyn N.Y. (1950).

10. G.W. Scott-Blair, *Survey of General and Applied Rheology*, Pitman, London (1949).
11. M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II., *Geophys. J. R. Astr. Soc.*, **13** (1967), 529-539.
12. M. Caputo, *Elasticità e Dissipazione*, Zanichelli, Bologna (1969).
13. M. Caputo & F. Mainardi, Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento* (Ser. II) **1** (1971), 161-198.
14. B. Gross, On creep and relaxation, *J. Appl. Phys.* **18** (1947), 212-221.
15. Y. N. Rabotnov, Equilibrium of an elastic medium with after effect, *Prikl. Matem. i Mekh.* **12** (1948), 81-91. [in Russian]
16. Y. N. Rabotnov, *Elements of Hereditary Solid Mechanics*, MIR, Moscow (1980). [English translation, revised from the 1977 Russian edition]
17. R. L. Bagley & P. J. Torvik, A theoretical basis for the application of fractional calculus, *J. Rheology* **27** (1983), 201-210.
18. R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, *J. Appl. Mech.* **51** (1984), 299-307.
19. T. F. Nonnenmacher & W. G. Glöckle, A fractional model for mechanical stress relaxation, *Phil. Mag. Lett.* **64** (1991), 89-93.
20. N. Heymans & J. -C. Bauwens, Fractal rheological models and fractional differential equation for viscoelastic behavior, *Rheol. Acta* **33** (1994), 210-219.
21. T. Pritz, Analysis of four-parameter fractional derivative model of real solid materials, *J. Sound and Vibr.* **195** (1996), 103-115.
22. V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics # 301, Longman, Harlow (1994).
23. V. Kiryakova, All the special functions are fractional differintegrals of elementary functions, *J. Physics A: Math.& Gen.* **30**, No 14 (1997), 5085-5103.
24. W. R. Schneider, The general solution of a non-linear integral equation of convolution type, *ZAMP* **33** (1982), 140-142.
25. W. R. Schneider & W. Wyss, Fractional diffusion and wave equations, *J. Math. Phys.* **30** (1989), 134-144.
26. U. J. Choi & R. C. MacCamy, Fractional order Volterra equations with applications to elasticity, *J. Math. Anal. Applics.* **139** (1989), 448-464.
27. A. N. Kochubei, A Cauchy problem for evolution equations of fractional order, *J. Diff. Eqs* **25** (1989), 967-974. [English transl. from Russian]
28. Y. Fujita, Integrodifferential equation which interpolates the heat equation and the wave equation, *Osaka J. Math.* **27** (1990), 309-321.
29. F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons & Fractals* **7** (1996), 1461-1477.

30. F. M a i n a r d i, The fundamental solutions for the fractional diffusion-wave equation, *Appl. Math. Lett.* **9**, No 6 (1996), 23-28.
31. F. M a i n a r d i & H. B u g g i s c h, On nonlinear waves in liquid-filled elastic tubes, in U. Nigul and J. Engelbrecht (Editors), *Nonlinear Deformation Waves*, pp. 87-100, Springer-Verlag, Berlin (1983). Proc. IUTAM Symposium, Tallinn, Estonia, 22-28 August 1982
32. R. R. N i g m a t u l l i n, The realization of the generalized transfer equation in a medium with fractal geometry, *Phys. Stat. Sol. B* **133** (1986), 425-430. [English transl. from Russian]
33. W. R. Y o u n g, A. P u m i r & Y. P o m e a u, Anomalous diffusion of tracer in convection rolls, *Phys. Fluids A* **1** (1989), 462-469.
34. A. L e M é h a u t é, *Les Géometries Fractales*, Hermes, Paris (1990).
35. A. O u s t a l o u p, *La Commande CRONE*, Hermes, Paris (1990).
36. T. F. N o n n e n m a c h e r, Fractional integral and differential equations for a class of Levy-type probability densities, *J. Phys A: Math. Gen.* **23** (1990), L697-L700.
37. N. S u g i m o t o, Burgers equation with a fractional derivative; hereditary effects of nonlinear acoustic waves, *J. Fluid Mech.* **225** (1991), 631-653.
38. M. G i o n a & H . E. R o m a n, Fractional diffusion equation for transport phenomena in random media, *Physica A* **185** (1992), 82-97.
39. R. L e n o r m a n d, Use of fractional derivatives for fluid flow in heterogeneous media, *PRE-PRINT* (1992). [paper presented at the 3-rd European Conf. on the Mathematics of Oil Recovery, 17-19 June 1992, Delft, The Netherlands]
40. M. O c h m a n n & S. M a k a r o v, Representation of the absorption of nonlinear waves by fractional derivatives, *J. Acoust. Soc. Am.* **94** (1993), 3392-3399.
41. M. C a p u t o, The splitting of the seismic rays due to dispersion in the Earth's interior, *Rend. Fis. Acc. Lincei* (Ser. IX) **4** (1993), 279-286.
42. F. M a i n a r d i, Fractional diffusive waves in viscoelastic solids, in J.L. Wegner and F.R. Norwood (Editors), *Nonlinear Waves in Solids*, ASME book No AMR 137, Fairfield NJ (1995), pp. 93-97. [Proc. IUTAM Symposium, University of Victoria, Canada, 15-20 August 1993]

    Abstract in *Appl. Mech. Rev.* **46** No 12 (1993), 549.
43. G. M. Z a s l a v s k y, Fractional kinetic equation for Hamiltonian chaos, *Physica D* **76** (1994), 110-122.
44. A. C a r p i n t e r i & F. M a i n a r d i (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien and New York (1997).

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