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A LONG STANDING CONJECTURE FAILED?

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In the frames of the Second International Workshop "Transform Methods & Special Functions, Varna'96" a round table discussion **"Physical and Geometrical Meanings and Applications of the Fractional Calculus Operators"** took place. Provoked by some open problems in the field of Fractional Calculus, it was well attended and lasted more than 2 hours. *Among the participants were*:

I. Ali, B. Bonilla, P. Butzer, N. Bozhinov, H.-J. Glaeske, R. Gorenflo, I. Dimovski, E. M. Elabd, S. L. Kalla, S. Kempfle, A. Kilbas, Yong Chan Kim, V. Kiryakova, P. Konovski, F. Mainardi, W. Majcharzak, S. Owa, I. Podlubny, M. Rivero, P. Rusev, M. Saigo, H. Saitoh, B. Al-Saqabi, H. M. Srivastava, J. Stankiewicz, R. Tremblay, J. Trujillo, Vu Kim Tuan, A. Al-Zamel, etc.

Topics of discussion were, along with the very recent practical applications, *some open problems raised subsequently on the three international conferences* devoted especially to Fractional Calculus, that took place in: Univ. of New Haven (USA) in 1974; Ross Priory, Univ. of Strathclyde - Scotland (UK) in 1984; Nihon University, Tokyo (Japan) - in occasion of its 100th Anniversary in 1989. Each of the Proceedings of these meetings, [1],[2],[3], was ended by a section of such open problems and especially, the following pessimistic conjecture was repeated there, *originally stated by the late Prof. B. Ross*, initiator of the first such conference.

CONJECTURE. **Is there a geometrical representation of a fractional derivative? If not, can one prove that a graphical representation of a fractional derivative does NOT exist? ...The consensus of the experts ...is that there is, in general, NO geometrical interpretation of a derivative of fractional order ... It can be asked, however at least for a geometrical meaning or a physical phenomena that can be represented by means of equations involving a derivative of a particular order such as** $1/2$ **...**

A suggestion that the *negative parts of the above conjecture might be wrong*, is given by the recent largely exploited notions of **"Fractal Hausdorff Dimensions"** and **"Fractal Geometry"**.

For the first time, my personal attention to these topics was attracted while reading the paper [4] in Proceedings [3]. I was also intrigued by the Editor's comment at its end: *"Though the present paper seems to be remotely related to fractional calculus, the editor accepted it to insert in the Proceedings in anticipation of the appearance of certain relations with fractional calculus in the future"*. Some of these doubts I presented in the final part (§5.7. Miscellaneous) of [5].

• Shortly, the **Fractals** are geometrical objects: points, curves, planes, cubes, etc. but unlike more familiar objects, they cannot be "measured" by conventional methods. The notion was introduced by Mandelbrot [6] who discovered that many of the irregular shapes that make up the natural world are not just random but can result from simple formulas, based on certain shapes nesting inside others. His observation of the existence of a *"Geometry of Nature"* has developed into a new kind of geometry, the *Fractal Geometry*. Whereas classical geometry is used to describe the form and structure of man-made objects, the fractals are models of both physical and non-physical structures such as clouds, coastlines (like the Black Sea's coastline near this Workshop's place), plants, snowflakes, etc. and even the fluctuation of the stock markets. Recently, the fractal theory has been developed intensively towards applications in geometry, analysis, graph theory, etc. The complex fractal images, which can be generated from simple rules, now play a major role in the rendering and modelling of natural phenomena in computer graphics, see e.g. Barnsley [7], a series of papers of Lapidus, Sendov et al. like [8],[9].

The *question for a possible relation between the* **Fract***als and* **Fract***ional Calculus* (not only semantically, in the coincidence of the first five letters) seems rather actual but yet foggy. A fractal set consisting of one-dimensional parts in a two-dimensional setting, is adopted to have a dimension between 1 and 2, i.e. a *fractal dimension*. It is based on the idea of the Hausdorff "measure" of a compact region in an *n-dimensional space, where n can be a fraction*. Then, we need a constructive interpretation of a fractional measure. By analogy, can an *integral of order* $1\frac{1}{2}$ be considered as a measure of a body's volume in a $1\frac{1}{2}$-dimensional space?

Open problem 1. Since the integer-dimensional spaces are Cartesian products of 1-dimensional spaces (lines) with a continuum cardinality, can we construct a $1\frac{1}{2}$-dimensional space as a product of 1-dimensional and $\frac{1}{2}$-dimensional spaces and how? How to describe a set whose dimension is less than 1?

• Open problems arise also from the **possibility to define the fractional derivatives by means of generalizations of finite differences**. It is well known that for a n-times differentiable function, the following formula

$$f^{(n)}(x) = \lim_{h\to 0} \frac{(\Delta_h^n f)(x)}{h^n}, \quad n = 1, 2, \dots$$

holds, where $(\Delta_h^n f)(x)$ denotes the backward finite difference of integer order n. This equality can be used for an alternative definition of the fractional derivatives of order $\alpha > 0$, via the *generalized differences of fractional order*

$$(\Delta_h^\alpha f)(x) = \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x - kh), \quad \alpha > 0,$$

namely:

$$D^{\alpha} f(x) = \lim_{h \to +0} \frac{\left(\Delta_h^{\alpha} f\right)(x)}{h^{\alpha}}, \quad \alpha > 0,$$

the so-called *Grünwald-Letnikov fractional derivatives*, originated by Liouville (1835), Grünwald (1867), Letnikov (1868), Mikolas (1963), etc. Recently, this approach to the fractional calculus has again attracted the attention of mathematicians, due to Butzer and Westphal [10] and Butzer, Dychkoff, Gorlich and Stens [11]. Their approach, from the point of view of the contemporary theory of functions, relates several classical problems of fractional calculus with the problems of functional analysis and gives also a way of dealing with generalized functions. Further, the differences of fractional order have been used to introduce *moduli of continuity of fractional order*:

$$\omega_{\alpha}(f,h) = \omega_{\alpha,\chi}(f,h) = \sup_{|t|<h} \left\| \Delta_t^{\alpha} f \right\|_{\chi},$$

χ being a Banach space; *fractional averaged moduli of smoothness*; spaces of functions of generalized smoothness; Hausdorff approximations; etc. Related results can be seen in Sendov and Popov [12], Dryanov [13], etc.

OPEN PROBLEM 2. Find an *appropriate discrete definition of the generalized fractional derivatives* from [5], analogous to the above and based on some *"fractional multiorder differences"*, generalizing Δ_h^{α} and possibly expressed in terms of the G-functions.

• A positive answer if there might be **any relation between the notions of fractional derivatives and fractional dimensions**, is suggested by some *other results of the Bulgarian group around acad. B. Sendov*, see for example the paper of Takev [14]. There, the graph $G(f)$ of a continuous function is considered as a *fractal set* with Hausdorff dimension $D(G(f))$. Suppose also that the fractional derivative $D^{\alpha} f$ exists but $D^{\alpha+\varepsilon} f$ does not exist for any $\varepsilon > 0$. Then,

$$\omega_p(f;h) \leq C h^{\alpha} \left\| D^{\alpha} f \right\|_p \qquad \left(\frac{1}{\alpha} > p > 1\right);$$

$$D\left(G(f)\right) \leq 2 - \alpha, \qquad 0 < \alpha < 1,$$

where $\omega_p(f;h)$ denotes the average modulus of continuity and the following problem arises: If for Lipschitz functions of order α the "$\leq$" sign can be replaced by "$=$" in the latter inequality?

In view of the above-mentioned as well as having in mind some other recent papers and the comments during the present discussion, we would like to hope that **Fractal Geometry might give, at last, a counterexample to the statements** in the CONJECTURE, a long standing challenge to the fractional analysts.

We can even state a more general

OPEN PROBLEM 3. What about *some geometrical or physical meanings and applications of the generalized fractional integrals and derivatives* from [5], or of their particular cases, involving special functions in the kernels ?

• The answers to these and many other open problems seem to be hidden in our **Fractal World**.

• **The questions, posed in the** CONJECTURE **have found excellent answers** in many recent papers dealing with practical applications of the fractional calculus' operators (including integrals and derivatives of order 1/2). Partly, this can be seen in some papers published in this volume and in the previous one [15], like these of Gorenflo, Kempfle, Mainardi, Podlubny et al., and also papers referred to there. Many useful results and comments towards practical applications of the fractional calculus can be found in the *encyclopaedic handbook* [16]. Let us point out that the honour of making the *first application* of derivatives of arbitrary order belonged to Abel (1823), who gave a solution of an integral equation arising in the formulation of the tautochrone problem. His solution was so elegant, that may be, it motivated Liouville to give a logical definition of the fractional derivatives. Also, the *first book* [17], *entirely devoted to the topic of fractional calculus*, was inspired by the practical point of view of its authors K. Oldham and J. Spanier; many comments on [17] by other participants in the round table, are cited in the discussions below. Some recent projects and papers of K. Oldham are related to *applications in chemistry, stock markets* etc. As a final part of this volume, the Editors have invited his contribution by correspondence, presenting shortly some of these aspects.

• To give **some impression on the spirit during this round table discussion**, in what follows we are trying to reproduce, partly and not pretending for any exactness and completeness, **pieces of taken records of the sayings of some participants**. The abbreviation **FC** stands for "Fractional Calculus".

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• F. MAINARDI: I would like to consider again *the problem related to Fractional Calculus and Fractality. It seems to me, it is not a closed problem, not yet* ! There are people, like Nigmatullin (see for example [18]), who are very fond of this relation.

Any way, I would like to inform you that next September (1996) Prof. Carpinteri (Structural Mechanics) and I (Mathematical Physics) are organizing an international school (at the seat of CISM in Udine, Italy) with the title *Scaling Laws and Fractality in Continuum Mechanics* and subtitle *A Survey of the Methods Based on Renormalization Group and Fractional Calculus.* Our motivation is based on the fact that there is some general feeling that FC could play an important role in the fractal world. A distinguished lecturer will be Dr. Lenormand, (the French Institut of Petroleum), who will discuss the role played by fractals in porous media. He has proposed a transport equation of fractional order for these media.

It is not strange that several researchers have arrived at ordinary or partial fractional differential equations. In the book by Oldham and Spanier ([17]) the common diffusion equation is treated by splitting the temporal-spatial operator into a product of two operators both of which are of order $1/2$. So usually, when people ask me about the meaning of the fractional derivative, I reply by giving this example.

In the CISM course just mentioned, see [19], I will give only lectures about FC and its applications in continuum (solid and fluid) mechanics. Anyway, since the relation between fractals and FC is yet open, the lecturers are expected only to start the discussion on it. Hopefully, there will be young and promising researchers attending the course, who can find this relation. My actual opinion is that *if there is some relation, it should be only in the asymptotic sense*. We know that in the fractal world the scaling laws play a fundamental role. Scaling laws are based on power laws and these power laws can be easily derived from the asymptotic behaviour of Mittag-Leffler type functions which enter naturally in problems of FC, like exponential type functions – in the ordinary calculus. Furthermore, in my opinion, if some relation between fractals and FC exists, such a relation should not be in a strict sense, because the fractional operators are based on the concepts of continuity, while the fractal world is based on discontinuity.

• P. BUTZER: If you consider everything in the Walsh sense (*Walsh Calculus*), then you can also work in a noncontinuous sense, see e.g. works of my school. Do you think there is some connection? (Discussion by P. Butzer, F. Mainardi, R. Gorenflo).

• F. MAINARDI: I think that *to make propaganda of FC, it would be good to relate in some sense FC to Fractals*, because the topic of fractals, appeared in recent decades, is most popular. Usually physicists and engineers do not find it surprising to work with non-integer dimensions but show some perplexity to be acquainted with fractional calculus which allows integration and derivation of any order. On the other hand, mathematicians usually prefer to work in functional analysis and to be "embedded" in a functional space, so that they are led to reject fractional calculus, if it is not carried out with strict rigor. In view of the difficulties in convincing people to accept this calculus, any relation between FC and Fractals would be welcome!

• R. GORENFLO: But I think, nevertheless, that the *fractional differential operator of order* $1/2$ *arises very naturally in problems of mathematical physics*, like boundary value problems, e.g. in the Dirichlet-Neumann map which relates the boundary values of a solution to a second order elliptic or parabolic equation to its normal derivative. A typical problem, related to fractional derivative of order $1/2$, is also the problem of *spherical stereology* (see contribution of R.G. in the same volume). There are some applications with the Tricomi equation when not only operators of order $1/2$, but of arbitrary order could arise; this equation is important in *aerodynamics*. Also, fractional operators of order $1/2$ arise in a very natural way in *geometric probability theory* and in some *problems of tomography*, in particular in tomography of objects with circular or spherical symmetry.

Other important applications are to *physical processes with after-effects*. FC is very convenient, if the memory function is given by a power law, with exponent not necessarily $1/2$. Since often such power law models can be fully analyzed mathematically, they can serve as useful simple models for much more complicated processes, and studying them in detail gives a lot of insight.

• S. KALLA: I think that the natural sciences and engineering provide the real physical meaning, as well as the motivation, for fractional derivatives and integrals. Mathematics, on the other hand, develops new ideas and methods for tackling the original physical problems and, very often, transcends that objective in order to establish a more general theory. This *alliance between physical intuition and mathematical versatility* was clearly manisfested in the joint work of Oldham and Spanier [17], a chemist and a mathematician when they were treating problems of mass and heat transfer; the semi-derivatives and semi-integrals.

During the 2nd International Conference in Japan (see [2]) I had many discussions with R. Bagley, a mathematician-engineer at the Air Force Institute of Technology (Wright Patterson, Ohio, USA) working on viscoelasticity. He was telling me about *many processes that could not be described by ordinary derivatives, hence requiring the use of fractional derivatives and integrals*, but I must admit that at this moment, I am unaware of how they are used in this respect. I have also good friends in Italy (physicists) who have written excellent papers in "Radiation Physics and Chemistry" in which solutions of certain integral and differential equations of atomic physics are given in terms of hypergeometric functions by means of fractional calculus. Therefore, *it is essential that we direct our efforts in promoting the field of FC to the physics and engineering communities which will provide us with answers as to what it all means in physical terms.*

• A. KILBAS: In our book [16], there is a lot of information on these problems. Another information: this February (1996) in Minsk (Belarus), an international conference: "Boundary Value Problems, Fractional Calculus and Special Functions" took place. We will publish also the Proceedings of this conference, hope this September (1996). Now, I would like to cite some words of the Preface of another Proceedings ([3]): *The subject "Fractional Calculus and its Applications" is a very important field in mathematics, natural sciences and technology.*

In the second edition of our book [16], in English - 5 years later, due to the intensive development of the topic, we were to add more that 300 new citations, and I think, this field (of FC) is developed now in *4 main directions*: 1) Generalizations and modifications of FC - operators involving various special functions; 2) Multidimensional FC, connected to Riesz potentials; 3) Use of methods and techniques of FC in mathematical disciplines like differential and integral equations, theory of series, of functions and so on. 4) Applications of FC to some applied scientific areas, e.g.: physics, mechanics, chemistry, etc.

In connection to 3), I would like to tell you when the first differential equation of fractional order appeared: in 1919! Independently, L. O'Shaughnessy and E. L. Post found solutions of the equation $D^{1/2}y = y/x$, but these solutions were different, and they discussed "why?" It turned out they had used different definitions of the fractional derivative $D^{1/2}$, practically what we denote now by $D_{+}^{1/2}$ and $D_{-}^{1/2}$ ...

It seems Professors Mainardi and Gorenflo told us many things about direction 4). Indeed, there are many practical problems where fractional operators, and especially - of order $1/2$, naturally appear. You can see such examples in the book [17] by Oldham (who is a specialist in electrochemistry) and Spanier (who is a specialist in mathematics)!

I think, at present, *if there is some connection between FC and the Fractals - which are more popular now ... But not so many mathematicians who work in FC* *know about Fractals, and who work in Fractals do not know what is FC; so if they knew - they could obtain some new results.*

• P. BUTZER: I am glad to know about many papers by the school from Minsk (about a dozen papers) ... Especially, the book of Samko, Kilbas, Marichev [16] which was not familiar to me because we work in different directions, so that we could not follow all of the various fields ...

But to my mind, it is very natural way *to define first the fractional derivatives via fractional differences which are infinite series, and then, just as in my lecture this morning, one can define the integrals.* So, the two approaches are inverse in that sense, the mentioned - which is different from the classical one, when you start from fractional integration and then going to fractional differentiation.

Some of these different approaches to define FC are more suitable when you work in spaces of periodic functions, Fourier transforms on the whole line, or when you work on the semi-axis using the Laplace transform ... So there are many different approaches ... If one works via fractional differences ... (Discussion: V. Kiryakova, A. Kilbas, P. Butzer).

• J. TRUJILLO: The observation of Prof. Kilbas is very good, it is very useful to give different definitions of fractional derivatives. Also, I would like to say I am highly interested in this field and I liked also Prof. Mainardi's comments.

But when applying fractional derivatives, people must be very careful, because for example, $D^{1/2}D^{1/2}$ is not always equal to D^1, so it is very useful to know and to prove under which conditions this is true. You can work only when it is true ... Miller has introduced such interesting concept ... He works with solutions of fractional differential equations only when the above relation is true, only for such functions.

• S. KEMPFLE: Let me tell you a story. I was asked by a professor in mechanics, about 5 years ago, to solve an *equation of fractional damping law* and this was my first connection with the fractional derivatives.I was surprised by the number of ways to defining them ... He used Laplace transform, because he was familiar with it, and then we tried by the techniques of functional analysis to define the involved operators, not only the derivative. We did not give any explicit definitions, we were to fix the function space to be successful, and we obtained a very good operator with all the good properties ... But we could not define the single derivative: we only could define the whole operator. Because we had no solution of the homogeneous equation, we mainly investigated the question of *causality*. And the results of this damping law were very good and useful.

• I. DIMOVSKI: The history of FC is going back to Leibnitz and his answer to the question "What about differential of order 1/2?" but I still do not know what it is ... But I would like to comment that the derivative of order 1/2 appears for example, in the Dirac equation in the *relativistic quantum mechanics*, and I wonder if somebody could say what this means? Really, from the Dirac equation one could obtain as a consequence, the Klein-Gordon equation, but they are by no means equivalent, since the Dirac equation is of 1st order and the Klein-Gordon's - of

2nd, so in order to consider some problem for the Klein-Gordon equation, we need *additional boundary value conditions which have no physical meaning* (?), so ... I am speaking this as a dilettante, but there is a Bulgarian theoretical physicist (Christo Dimitrov), who tried *to explain some physical phenomena by direct treating of such differential equations of half-order* ...

• I. PODLUBNY: When I have to explain to my colleagues (who are engineers) the meaning of the *left-sided* Riemann-Liouville fractional integral (FI), I use its interpretation as a cumulated impact of the *past* of the dynamical process $f(\tau)$ on its state at the time $\tau = t$. If $f(\tau)$ represents the change of some parameter in time (*history of events*), then the Abel kernel $A(t-\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$ represents the weight assigned to a particular *event* $f(\tau)$ to reflect its impact on the state of the process at the time $\tau = t$, and the integral $\int_a^t A(t-\tau)f(\tau)d\tau$ gives the cumulated impact of all previous states of the process on its present state. The Grünwald-Letnikov definition can be interpreted in a similar way. Such an interpretation of FI also explains why the term *terminals* (*departures* and *arrivals*), introduced by Prof. B. Ross for denoting the limits of fractional integration and differentiation, is natural and acceptable: the *trajectory* of the process $f(\tau)$ *departs* from $\tau = a$ and *arrives* at $\tau = t$ ...

Is it possible to interpret the *right-sided* FI, $\int_t^b \dots d\tau$, as the cumulated impact of the *future* of the process $f(\tau)$ on its present state? I think the answer is *yes*, because *it is not only our past which determines our present behavior, but also the predictable future development*. In the process control, we can take into account the predictable development of the state of a system using the right-sided FI-s. In my opinion, left-sided and right-sided FI-s and fractional derivatives may be used for the feed-back and feed-forward control, correspondingly.

Regarding possible links between FC and Fractals ... First, at present I think that it will be difficult to establish a clear link, if we consider *static* fractal objects. On the other hand, several authors, for example Prof. A. Oustaloup and Prof. R. Nigmatullin, arrived at fractional differential equations when they modelled *dynamical processes in fractals*. Second, fractal structures, like binary trees, are now often used for developing new *physical* models – for example, in viscoelasticity; there are also works in which electric circuits of the fractal structure are considered. In all such cases, the corresponding *mathematical* models are also of fractional order.

Among our results related to FC and Fractals I would like to mention two. First, a *concept of the $PI^\lambda D^\mu$ controller*. Instead of "jumping" between classical controllers of the type P ($\lambda = \mu = 0$), PI ($\lambda = 1$, $\mu = 0$), PD ($\lambda = 0$, $\mu = 1$), and PID ($\lambda = \mu = 1$), we can now move in the (λ, μ)-plane continuously and achieve better control. Technically, a $PI^\lambda D^\mu$ controller can be made using circuits of self-similar structure, suggested, for example, by Prof. K. B. Oldham. Another approach to the application of FC in control (*CRONE control*) was suggested by Prof. A. Oustaloup earlier. Second, the fact that the plots representing the change of concentration of O_2 and CO in a blast furnace have non-integer dimension $D > 1$, probably due to fractality of coke and ore. This indicates that a *more adequate mathematical model of heat and mass exchange in a blast furnace may be of fractional order.*

• F. Mainardi: *To convince people that FC is useful, it is only a question of exciting their own mentality.* For a physicist it is important to "see" some physical meaning, or, at least, something related to a familiar concept of physics. I would like to take the words of comment by S. Kalla about the paper of P. Butzer, that Mellin transform is not only useful but also beautiful ... I would like to say these words also for FC: *FC is not only useful but it is also beautiful* ... (Comment by V. Kiryakova: *Or not only beautiful but also useful?*) ... Eh, this depends on the point of view ... For me the "continuity" is "beautiful" and FC is beautiful since it allows *to save the continuity between some physical phenomena or mathematical properties* ... For example, the interpolation of factorials by the Gamma-functions: one can interpolate $n!$ in many ways, but only this one is so "beautiful" ... You can interpolate models of relaxation - oscillation and diffusion - wave propagation in several manners but the FC seems the only one that provides the *possibility of continuous transition between well-known phenomena* in a rather simple manner from a mathematical point of view ...

• P. Rusev: It seems that the constant 1/2 plays some essential role in mathematics? Let me mention the classical Riemann hypothesis that the nontrivial zeros of the Riemann zeta-functions are connected with this constant ...

• H.M. Srivastava: As A. Kilbas stressed, I am one who attended all the mentioned three conferences on FC. At the third one, in Japan, I was asked to edit the Open Questions Session and this publication in [3] is written by me. We concentrated here mainly on one of this open problems, but I would like to mention and to ask what has happened to the others? Let me cite these open problems again: ... (The open problems raised in Japan in 1989, see [3], pp. 281-284, have been read one by one ...). Probably Kilbas could tell us more about what has happened?

• V. Kiryakova: In the book of Samko, Kilbas, Marichev [16] there is a special section discussing the fate of these open problems ... Also, let me emphasize that most of these open questions could be posed and would be much more complicated when generalized fractional integrals and derivatives (like in [5],[20]) are considered.

• A. Kilbas cited the answers given in [16]: 5 of the open questions have been already answered, but 7 others are still open.

• V. Kiryakova: Then, do you feel that a number of 7 next Workshops would be enough to give answers to these problems? Can we make the schedule of the next meetings on FC right now? It would be welcome if we can collect some written notes on this round table discussion ....

• Further, the participants discussed: *the need of specialized publications on the topic, the possible next meetings on FC*, where they could take place and from where sponsorship could be looked for?

References

[1] B. R o s s (ed.), *Fractional Calculus and Its Applications (Proc. Internat. Conf. Held in New Haven, 1974).* L. N. M. **457**, Springer-Verlag, N. York (1975).

[2] A. M c B r i d e, G. R o a c h (ed-s), *Fractional Calculus (Proc. Internat. Workshop at Ross Priory, Univ. of Strathclyde, 1984).* Res. N. M. **138**, Pitman, Boston-London-Melbourne (1985).

[3] K. N i s h i m o t o (ed.), *Fractional Calculus and Its Applications (Proc. Internat. Conf. Held at Nihon Univ., Tokyo, 1989).* Nihon Univ., Koriyama (1990).

[4] S. K u y o s y i n g, Fractals: A new dimension in mathematics. In: [3], 270-280.

[5] V. K i r y a k o v a, *Generalized Fractional Calculus and Applications.* Longman, Harlow (co-publ. John Wiley & Sons, N. York) (1994).

[6] B. B. M a n d e l b r o t, *The Fractal Geometry of Nature.* W. H. Freeman and Co, N. York (1983).

[7] M. B a r n s l e y, *Fractals Everywhere.* Acad. Press, San Diego (1988).

[8] M. L a p i d u s, Can one hear the shape of a fractal drum? Partial resolution of the Weyl-Berry conjecture. *Geom. Anal. and Computer Graphics* (MSRI, Berkeley, 1988) **17** (1991), 119-126.

[9] B. S e n d o v, P. M a r i n o v, Orthonormal systems of fractal functions. *Math. Balkanica (N.S.)* **11**, No 1-2 (1997), 169-200.

[10] P. B u t z e r, U. W e s t p h a l, An access to fractional differentiation via fractional difference quotiens. In: [1], 115-145.

[11] P. B u t z e r, H. D y c h k o f f, E. G ö r l i c h, R. S t e n s, Best trigonometric approximation, fractional order derivatives and Lipschitz classes. *Canad. J. Math.* **29** (1977), 781-793.

[12] B. S e n d o v, V. P o p o v, *Averaged Moduli of Smoothness.* Wiley Intersci. Publ. Ser., Chichester, N. York, etc. (1988).

[13] D. D r y a n o v, Average modulus of smoothness of fractional index and fractional order derivatives. *C. R. Acad. Bulg. Sci.* **35**, No 12 (1982), 1635-1637.

[14] M. T a k e v, On the Hausdorff dimension of completed graphs. *C. R. Acad. Bulg. Sci.* **45**, No 7 (1992), 13-15.

[15] P. R u s e v, I. D i m o v s k i, V. K i r y a k o v a (ed-s), *Transform Methods & Special Functions, Sofia'94 (Proc. 1st Internat. Workshop).* SCTP, Singapore (1995).

[16] S. S a m k o, A. K i l b a s, O. M a r i c h e v, *Integrals and Derivatives of Fractional Order and Some of Their Applications.* In Russian: Nauka I Tekhnika, Minsk (1987); Engl. transl.: Gordon & Breach, N. York etc. (1993).

[17] K. O l d h a m, J. S p a n i e r, *The Fractional Calculus.* Acad. Press, N. York (1974).

[18] R. N i g m a t u l l i n, The fractional integral and its physical interpretation. *Theor. and Math. Phys.* **90** (1992), 242-251.

[19] A. C a r p i n t e r i, F. M a i n a r d i (ed-s), *Fractals and Fractional Calculus in Continuum Mechanics.* Springer-Verlag, Wien and N. York (1997).

[20] V. K i r y a k o v a, Generalized fractional calculus, special functions and integral transforms. In: [15], 123-154.