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AFTERTHOUGHTS ON INTERPRETATION OF FRACTIONAL DERIVATIVES AND INTEGRALS

(Contribution to the Round Table discussion)

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The attempts (see [Ni]) to interpret or consider fractional integrals as generated by a convolution with a generalized function, nonvanishing only on a fractal nowhere dense set (actually, a Cantor set), are doomed to failure. This is an immediate consequence of Titchmarsh's theorem for convolutions, but can also be seen by other arguments. See [Ru1] and [Ru2].

Are there meaningful geometrical or physical interpretations? At least there are several applications of half-order integration or differentiation in analysis of rotationally symmetric configurations for measuring densities or other intensive quantities by the aid of integrals over straight lines, in some instances also over curved lines and subsequent appropriate change of variables. See [GV] and [CB] for examples in spectroscopic measurements, rotationally symmetric tomography (e.g. of globular clusters of stars in the sky, of Sol's corona or of planetary atmospheres), determination of the refraction index of optical fibres, inversion of seismic travel times. In spherical stereology (see [Go]) the probability density of the radius of a sphere is related to that of the radius of a circle obtained by a random plane cut via an Abel integral equation of order 1/2. And likewise the probability density of the length of a random straight line cutting a circle with the probability density of the radius of the circle. These aspects are in a very lucid way made clear in the paper [Re] by Reid. Basically, in these applications the order 1/2 arises directly via the Pythagoras theorem.

Applications with order 1/2 can also be found in Newtonian mechanics, the order 1/2 being caused by the fact that in Hamiltonian systems kinetic and potential energy are quadratic forms. Here Abel's isochrone and tautochrone problems are classical, see [Ab]. Another well known source of fractional or related operators, often again of order 1/2, is the theory of boundary value problems in partial

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differential equations. Let us only mention the Dirichlet-Neumann map relating boundary values of a solution of an elliptic or parabolic problem to its normal derivative at the boundary. Some problems of this type are discussed in [GW] where also for an interface problem for the Tricomi equation orders different from 1/2 occur.

In [Ru2] there is presented a detailed discussion of a relation between fractional integral equations and system theory. It is shown, among other things, how the effect of a fractional integration operator, after a suitable change of the independent variable, can be approximated by a finite system of ordinary differential equations.

Yet another field is that of processes with memory. If the memory effect is caused by a power law then naturally Abel-type integral or integro-differential equations arise, and aside from particular models there is no intrinsic reason for the order to be 1/2. Indeed, this field deserves to be studied in detail and in depth, the bonus for the general theory of evolution processes with memory being that for linear power law processes the theory of generalized Mittag-Leffler functions is available (see [MG]). Such power law processes can serve as valuable models of more general evolution processes with memory (see [Pr]). It is important to note that the fixation of initial values is distinct from that of most literature of pure mathematical character hitherto published in the field of fractional differential equations. This aspect is an instance of the beneficial influence of inspiration from outside of Mathematics.

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