FRACALMO PRE-PRINT 0201
Fractional Calculus and Applied Analysis, Vol. 2 No 4 (1999) 383-414.
http://www.math.bas.bg/~fcaa/

ANALYTICAL PROPERTIES AND APPLICATIONS OF THE WRIGHT FUNCTION *

Rudolf Gorenflo [1], Yuri Luchko [2], Francesco Mainardi [3]

Abstract

The entire function (of z)

$$\phi(\rho,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\rho k+\beta)},\ \rho>-1,\ \beta\in\mathbf{C},$$

named after the British mathematician E.M. Wright, has appeared for the first time in the case $\rho>0$ in connection with his investigations in the asymptotic theory of partitions. Later on, it has found many other applications, first of all, in the Mikusiński operational calculus and in the theory of integral transforms of Hankel type. Recently this function has appeared in papers related to partial differential equations of fractional order. Considering the boundary-value problems for the fractional diffusion-wave equation, i.e., the linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α with $0<\alpha\leq 2$, it was found that the corresponding Green functions can be represented in terms of the Wright function. Furthermore, extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright and the generalized Wright functions. In this *survey paper* we consider some of the above mentioned applications of the Wright function with special emphasis of its key role in the partial differential equations of fractional order.

* Partially supported by the the Research Commission of Free University of Berlin (Project "Convolutions") and by the Italian CNR and INFN. Paper presented at the 3rd Workshop TMSF-AUBG'99, http://www.math.bas/~tmsf/

We also give some analytical tools for working with this function. Beginning with the classical results of Wright about the asymptotics of this function, we present other properties, including its representations in terms of the special functions of the hypergeometric type and the Laplace transform pairs related to the Wright function. Finally, we discuss recent results about distribution of zeros of the Wright function, its order, type and indicator function, showing that this function is an entire function of completely regular growth for every $\rho > -1$.

Mathematics Subject Classification: primary 33E20, secondary 33C20, 30C15, 30D15, 26A33, 45J05, 45K05

Key Words and Phrases: Wright function, indicator function, asymptotics of zeros, generalized hypergeometric function, diffusion-wave equation, Green function, scale-invariant solutions, Erdélyi-Kober operators

1. Introduction

The purpose of this survey paper is to outline the fundamental role of the Wright function in partial differential equations of fractional order, to consider some other applications of this function and to give its analytical properties including asymptotics and distribution of its zeros. Partial differential equations of fractional order (FPDE) are obtained by replacing some (or all) derivatives in partial differential equations by derivatives of fractional order (in Caputo, Riemann-Liouville or inverse Riesz potential sense). Mathematical aspects of the boundary-value problems for some FPDE have been treated in papers by several authors including Engler [5], Fujita [7], Gorenflo and Mainardi [11], Mainardi [20]-[23], Podlubny [32], Prüss [34], Saichev and Zaslavsky [35], Samko et al. [36], Schneider and Wyss [37] and by Wyss [45].

¿From the other side, some FPDE were successfully used for modelling relevant physical processes (see, for example Giona and Roman [9], Hilfer [15], Mainardi [21], Metzler et al. [26], Nigmatullin [28], Pipkin [31], Podlubny [32] and references there). In applications, special types of solutions, which are invariant under some subgroup of the full symmetry group of the given equation (or for a system of equations) are especially important.

Recently, the scale-invariant solutions for time-fractional diffusion-wave equation (with the fractional derivative in the Riemann-Liouville sense) and for the more general time- and space-fractional partial differential equation (with the Riemann-Liouville space-fractional derivative of order $\beta \leq 2$ instead of the second order space derivative) have been presented by Buckwar and Luchko [1] and by Luchko and Gorenflo [19], respectively. The case of the time-fractional diffusion-wave equation with the Caputo fractional derivative has been considered by Gorenflo, Luchko and Mainardi [13].

The plan of the paper is as follows. In Section 2, following the papers by Djrbashian and Bagian [4], Gajić and Stanković [8], Luchko and Gorenflo [19],

Mainardi [23], Mainardi and Tomirotti [24], Mikusiński [27], Pathak [30], Pollard [33], Stanković [39], and Wright [42], [44], we recall the main properties of the Wright function including its integral representations, asymptotics, representations in terms of the special functions of the hypergeometric type and the Laplace transform pairs related to the Wright function. Finally, we discuss new results about distribution of zeros of the Wright function, its order, type and indicator function, showing that this function is an entire function of completely regular growth for every $\rho > -1$.

In Section 3, we outline some applications of the Wright function, beginning with the results by Wright [41] in the asymptotic theory of partitions. Special attention is given to the key role of the Wright function in the theory of FPDE. Following Gorenflo, Mainardi and Srivastava [14] and Mainardi [20]-[23], we consider in details the boundary-value problems of Cauchy and signalling type for the fractional diffusion-wave equation, showing that the corresponding Green functions can be represented in terms of the Wright function. We present also some results from Buckwar and Luchko [1], Gorenflo, Luchko and Mainardi [13], Luchko and Gorenflo [19] concerning the extension of the methods of the Lie groups in partial differential equations to FPDE. It will be shown that some of the group-invariant solutions of FPDE can be given in terms of the Wright and the generalized Wright functions.

We remark finally that the present review is essentially based on our original works. For the other applications of the Wright function, including Mikusiński's operational calculus and the theory of integral transforms of Hankel type we refer, for example, to Kiryakova [16], Krätzel [17], Mikusiński [27], and Stanković [39].

2. Analytical properties

2.1. Asymptotics

Probably the most important characteristic of a special function is its asymptotics. In the case of an entire function there are deep relations between its asymptotic behaviour in the neighbourhood of its only singular point – the essential singularity at $z = \infty$ – and other properties of this function, including distribution of its zeros (see, for example, Evgrafov [6], Levin [18]). It follows from the Stirling asymptotic formula for the gamma function that the Wright function

$$\phi(\rho, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \beta)}, \ \rho > -1, \ \beta \in \mathbf{C}, \tag{1}$$

is an entire function of z for $\rho > -1$ and, consequently, as we will see in the later parts of our survey, some elements of the general theory of entire functions can be applied.

The complete picture of the asymptotic behaviour of the Wright function for large values of z was given by Wright [42] in the case $\rho > 0$ and by Wright [44] in the case $-1 < \rho < 0$. In both cases he used the method of steepest descent and the integral representation

$$\phi(\rho, \beta; z) = \frac{1}{2\pi i} \int_{\mathrm{Ha}} e^{\zeta + z\zeta^{-\rho}} \zeta^{-\beta}\, d\zeta, \ \rho > -1, \ \beta \in \mathbf{C} \tag{2}$$

where Ha denotes the Hankel path in the ζ-plane with a cut along the negative real semi-axis $\arg \zeta = \pi$. Formula (2) is obtained by substituting the Hankel representation for the reciprocal of the gamma function

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{\mathrm{Ha}} e^{\zeta} \zeta^{-s}\, d\zeta, \ s \in \mathbf{C} \tag{3}$$

for $s = \rho k + \beta$ into (1) and changing the order of integration and summation.

Let us consider at first the case $\rho > 0$.

THEOREM 2.1.1. *If* $\rho > 0, \ \arg(-z) = \xi, \ |\xi| \le \pi$, *and*

$$Z_1 = (\rho|z|)^{1/(\rho+1)} e^{i(\xi+\pi)/(\rho+1)}, \ Z_2 = (\rho|z|)^{1/(\rho+1)} e^{i(\xi-\pi)/(\rho+1)},$$

then we have

$$\phi(\rho, \beta; z) = H(Z_1) + H(Z_2), \tag{4}$$

where $H(Z)$ *is given by*

$$H(Z) = Z^{\frac{1}{2}-\beta} e^{\frac{1+\rho}{\rho} Z} \left\{ \sum_{m=0}^{M} \frac{(-1)^m a_m}{Z^m} + \mathrm{O}\left(\frac{1}{|Z|^{M+1}} \right) \right\}, \ Z \to \infty \tag{5}$$

and the $a_m, \ m = 0, 1, \ldots$, *are defined as the coefficients of* v^{2m} *in the expansion of*

$$\frac{\Gamma(m + \frac{1}{2})}{2\pi} \left(\frac{2}{\rho + 1} \right)^{m+\frac{1}{2}} (1 - v)^{-\beta} \{g(v)\}^{-2m-1}$$

with

$$g(v) = \left\{ 1 + \frac{\rho + 2}{3} v + \frac{(\rho + 2)(\rho + 3)}{3 \cdot 4} v^2 + \ldots \right\}^{\frac{1}{2}}.$$

In particular, if $\beta \in \mathbf{R}$ we get the asymptotic expansion of the Wright function $\phi(\rho, \beta; -x)$ for $x \to +\infty$ in the form

$$\phi(\rho, \beta; -x) = x^{p(\frac{1}{2}-\beta)} e^{\sigma x^p \cos \pi p} \cos\left(\pi p (\frac{1}{2} - \beta) + \sigma x^p \sin \pi p \right) \left\{ c_1 + O(x^{-p}) \right\}, \tag{6}$$

where $p = \frac{1}{1+\rho}$, $\sigma = (1 + \rho)\rho^{-\frac{\rho}{1+\rho}}$ and the constant c_1 can be exactly evaluated.

If we exclude from the consideration an arbitrary small angle containing the negative real semi-axis, we get a simpler result.

THEOREM 2.1.2. *If* $\rho > 0,\ \arg z = \theta,\ |\theta| \le \pi - \epsilon,\ \epsilon > 0$, *and*

$$Z = (\rho|z|)^{1/(\rho+1)} e^{i\theta/(\rho+1)},$$

then we have

$$\phi(\rho, \beta; z) = H(Z), \tag{7}$$

where $H(z)$ *is given by* (5).

In the case $\rho = 0$ the Wright function is reduced to the exponential function with the constant factor $1/\Gamma(\beta)$:

$$\phi(0, \beta; z) = \exp(z)/\Gamma(\beta), \tag{8}$$

which turns out to vanish identically for $\beta = -n,\ n = 0, 1, \ldots$.

To formulate the results for the case $-1 < \rho < 0$ we introduce some notations. Let

$$y = -z,\ -\pi < \arg z \le \pi,\ -\pi < \arg y \le \pi, \tag{9}$$

and let

$$Y = (1+\rho)\left((-\rho)^{-\rho} y\right)^{1/(1+\rho)}. \tag{10}$$

THEOREM 2.1.3. *If* $-1 < \rho < 0,\ |\arg y| \le \min\{\frac{3}{2}\pi(1+\rho), \pi\} - \epsilon,\ \epsilon > 0$, *then*

$$\phi(\rho, \beta; z) = I(Y), \tag{11}$$

where

$$I(Y) = Y^{\frac{1}{2}-\beta} e^{-Y} \left\{ \sum_{m=0}^{M-1} A_m Y^{-m} + \mathrm{O}(Y^{-M}) \right\},\ Y \to \infty, \tag{12}$$

and the coefficients $A_m,\ m = 0, 1 \ldots$ *are defined by the asymptotic expansion*

$$\begin{aligned} \frac{\Gamma(1-\beta-\rho t)}{2\pi(-\rho)^{-\rho t}(1+\rho)^{(1+\rho)(t+1)}\Gamma(t+1)} &= \sum_{m=0}^{M-1} \frac{(-1)^m A_m}{\Gamma((1+\rho)t + \beta + \frac{1}{2} + m)} \\ &\quad + \mathrm{O}\left(\frac{1}{\Gamma((1+\rho)t + \beta + \frac{1}{2} + M)} \right), \end{aligned}$$

valid for $\arg t,\ \arg(-\rho t)$, *and* $\arg(1-\beta-\rho t)$ *all lying between* $-\pi$ *and* π *and* t *tending to infinity.*

If $-1/3 \le \rho < 0$, the only region not covered by Theorem 2.1.3 is the neighbourhood of the positive real semi-axis. Here we have the following result.

THEOREM 2.1.4. *If* $-1/3 < \rho < 0,\ |\arg z| \le \pi(1+\rho) - \epsilon,\ \epsilon > 0$, *then*

$$\phi(\rho, \beta; z) = I(Y_1) + I(Y_2), \tag{13}$$

where $I(Y)$ *is defined by* (12),

$$Y_1 = (1+\rho)\left((-\rho)^{-\rho} z e^{\pi i}\right)^{1/(1+\rho)}, Y_2 = (1+\rho)\left((-\rho)^{-\rho} z e^{-\pi i}\right)^{1/(1+\rho)}, \tag{14}$$

hence

$$Y_1 = Y \text{ if } -\pi < \arg z \le 0, \quad \text{and} \quad Y_2 = Y \text{ if } 0 < \arg z \le \pi.$$

As a consequence we get the asymptotic expansion of the Wright function $\phi(\rho, \beta; x)$ for $x \to +\infty$ in the case $-1/3 < \rho < 0,\ \beta \in \mathbf{R}$ in the form:

$$\phi(\rho, \beta; x) = x^{p(\frac{1}{2}-\beta)} e^{-\sigma x^p \cos \pi p} \cos(\pi p(\frac{1}{2} - \beta) - \sigma x^p \sin \pi p) \left\{c_2 + O(x^{-p})\right\}, \tag{15}$$

where $p = \frac{1}{1+\rho}$, $\sigma = (1+\rho)(-\rho)^{-\frac{\rho}{1+\rho}}$ and the constant c_2 can be exactly evaluated. When $-1 < \rho < -1/3$, there is a region of the plane in which the expansion is algebraic.

THEOREM 2.1.5. *If* $-1 < \rho < -1/3,\ |\arg z| \le \frac{1}{2}\pi(-1-3\rho) - \epsilon,\ \epsilon > 0$, *then*

$$\phi(\rho, \beta; z) = J(z),\ z \to \infty, \tag{16}$$

where

$$J(z) = \sum_{m=0}^{M-1} \frac{z^{(\beta-1-m)/(-\rho)}}{(-\rho)\Gamma(m+1)\Gamma(1+(\beta-m-1)/(-\rho))} + \mathrm{O}(z^{\frac{\beta-1-M}{-\rho}}). \tag{17}$$

Finally, the asymptotic expansions of the Wright function in the neighbourhood of the positive real semi-axis in the case $\rho = -1/3$ and in the neighbourhood of the lines $\arg z = \pm\frac{1}{2}\pi(-1-3\rho)$ when $-1 < \rho < -1/3$ are given by the following results by Wright.

THEOREM 2.1.6. *If* $\rho = -1/3,\ |\arg z| \le \pi(1+\rho) - \epsilon,\ \epsilon > 0$, *then*

$$\phi(\rho, \beta; z) = I(Y_1) + I(Y_2) + J(z), \tag{18}$$

where $I(Y)$ *is defined by* (12), Y_1, Y_2 *by* (14), *and* $J(z)$ *by* (17).

THEOREM 2.1.7. *If* $-1 < \rho < -1/3,\ |\arg z \pm \frac{1}{2}\pi(-1-3\rho)| \le \pi(1+\rho) - \epsilon,\ \epsilon > 0$, *then*

$$\phi(\rho, \beta; z) = I(Y) + J(z), \tag{19}$$

where $I(Y)$ *is defined by* (12) *and* $J(z)$ *by* (17).

The results given above contain the complete description of the asymptotic behaviour of the Wright function for large values of z and for all values of the parameters $\rho > -1,\ \beta \in \mathbf{C}$. We will use them repeatedly in our further discussions.

2.2. Representations through hypergeometric functions

Due to the relation

$$\phi(1,\nu+1;-\frac{1}{4}z^2)=\left(\frac{z}{2}\right)^{-\nu}J_\nu(z). \tag{20}$$

Wright considered the function $\phi(\rho,\beta;z)$ as a generalization of the Bessel function $J_\nu(z)$. In the general case of arbitrary real $\rho>-1$ the Wright function is a particular case of the Fox H-function ([13], [14], [16, App. E], [38, Chapter 1]):

$$\phi(\rho,\beta;z)=H_{0,2}^{1,0}\left[-z\,\middle|\,\begin{matrix}-\\(0,1),(1-\mu,\rho)\end{matrix}\right]. \tag{21}$$

Unfortunately, since the Fox H-function is a very general object this representation is not especially informative. It turns out that if ρ is a positive rational number the Wright function can be represented in terms of the more familiar generalized hypergeometric functions. Let be $\rho=n/m$ with positive integers n and m. Substituting $s=ms_1$ into (21) and making use of the Gauss-Legendre formula for the gamma function

$$\Gamma(nz)=n^{nz-\frac{1}{2}}(2\pi)^{\frac{1-n}{2}}\prod_{k=0}^{n-1}\Gamma(z+\frac{k}{n}),\quad n=2,3,\dots$$

we arrive at the representation

$$\phi(\frac{n}{m},\beta;z)=(2\pi)^{\frac{n-m}{2}}m^{\frac{1}{2}}n^{-\beta+\frac{1}{2}}\frac{1}{2\pi i}\int\limits_{\mathrm{L}_{-\infty}}\frac{\prod\limits_{k=0}^{m-1}\Gamma(s_1+\frac{k}{m})}{\prod\limits_{l=0}^{n-1}\Gamma(\frac{\beta}{n}-s_1+\frac{l}{n})}\left(\frac{(-z)^m}{m^m n^n}\right)^{-s_1}ds_1, \tag{22}$$

which is equivalent to the representation given by Pathak [30] in terms of the Meijer G-function ([16, App. A], [25, Chapter 4]). Here $\mathrm{L}_{-\infty}$ is a loop beginning and ending at $-\infty$, encircling in the positive direction all the poles of $\Gamma(s_1+\frac{k}{m})$, $k=0,\dots,m-1$, i.e., the points $-\frac{k}{m}$, $-1-\frac{k}{m},\dots$. The residue theorem and the relation ([25, Chapter 3])

$$\mathrm{res}_{z=-k}\Gamma(z)=\frac{(-1)^k}{k!},\ k=0,1,2,\dots$$

allow us to represent this integral as a sum of m series of hypergeometric type:

$$\phi(\frac{n}{m},\beta;z)=(2\pi)^{\frac{n-m}{2}}\frac{m^{\frac{1}{2}}}{n^{\beta-\frac{1}{2}}}\sum_{p=0}^{m-1}\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\frac{\prod\limits_{\substack{k=0\\k\neq p}}^{m-1}\Gamma(-q-\frac{p-k}{m})}{\prod\limits_{l=0}^{n-1}\Gamma(\frac{\beta}{n}+\frac{p}{m}+q+\frac{l}{n})}\left(\frac{(-z)^m}{m^m n^n}\right)^{q+\frac{p}{m}}.$$

Using the Gauss-Legendre formula and the recurrence and reflection formulae for the gamma function

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

to simplify the coefficients of the series in the last representation we obtain the final formula

$$\phi(\frac{n}{m}, \beta; z) = \sum_{p=0}^{m-1} \frac{z^p}{p!\Gamma(\beta + \frac{n}{m}p)}\, {}_0F_{n+m-1}\left(-; \Delta(n, \frac{\beta}{n} + \frac{p}{m}), \Delta^*(m, \frac{p+1}{m}); \frac{z^m}{m^m n^n}\right), \tag{23}$$

where ${}_pF_q((a)_p; (b)_q; z)$ is the generalized hypergeometric function ([25, Chapter 4]),

$$\Delta(k, a) = \{a, a + \frac{1}{k}, \ldots, a + \frac{k-1}{k}\}, \quad \Delta^*(k, a) = \Delta(k, a) \setminus \{1\}.$$

We note that the set $\Delta^*(k, a)$ is correctly defined in our case since the number 1 is an element of the set $\Delta(m, \frac{p+1}{m})$, $0 \le p \le m-1$.

The same considerations can be applied in the case of negative rational ρ but under the additional condition that the parameter β is also a rational number. In particular, we obtain the formulae

$$\phi(-\frac{1}{2}, -n; z) = \frac{(-1)^{n+1} z}{\pi} \Gamma(\frac{3}{2} + n)\ {}_1F_1(\frac{3}{2} + n; \frac{3}{2}; -\frac{z^2}{4}),\ n = 0, 1, 2, \ldots, \tag{24}$$

$$\phi(-\frac{1}{2}, \frac{1}{2} - n; z) = \frac{(-1)^n}{\pi} \Gamma(\frac{1}{2} + n)\ {}_1F_1(\frac{1}{2} + n; \frac{1}{2}; -\frac{z^2}{4}),\ n = 0, 1, 2, \ldots. \tag{25}$$

If $n = 0$ we get

$$\phi(-\frac{1}{2}, 0; z) = -\frac{z}{2\sqrt{\pi}} e^{-z^2/4}, \tag{26}$$

$$\phi(-\frac{1}{2}, \frac{1}{2}; z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}. \tag{27}$$

The formula (26) was given by Stanković [39]. He also gave the relation ($x > 0$)

$$\phi(-\frac{2}{3}, 0; -x^{-\frac{2}{3}}) = -\frac{1}{2\sqrt{3\pi}} \exp(-\frac{2}{27x^2}) W_{-\frac{1}{2}, \frac{1}{6}}\left(-\frac{4}{27x}\right),$$

where $W_{\mu,\nu}(x)$ is the Whittaker function satisfying the differential equation

$$\frac{d^2}{dx^2} W(x) + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{\nu^2}{4x^2}\right) W(x) = 0.$$

The formula (27) as well as some other particular cases of the Wright function with ρ and β rational, $-1 < \rho < 0$, can be found in Mainardi and Tomirotti [24], where a particular case of the Wright function, namely, the function

$$M(z; \beta) = \phi(-\beta, 1-\beta; -z),\ 0 < \beta < 1 \tag{28}$$

has been considered in details. For $\beta = 1/q,\ q = 2, 3, \ldots$ the representation

$$M(z; \frac{1}{m}) = \frac{1}{\pi} \sum_{n=1}^{m-1} (-1)^{n-1} \Gamma(n/m) \sin(\pi n/m) F(z; n, m) \tag{29}$$

with

$$F(z; n, m) = \sum_{k=0}^{\infty} (-1)^{k(m+1)} (n/m)_k \frac{z^{mk+n-1}}{(mk+n-1)!}$$

was given. Here $(a)_k,\ k = 0, 1, 2, \ldots$, denotes the Pochhammer symbol

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1).$$

In particular, the formula (29) gives us for $m = 3$ the representation

$$\phi(-\frac{1}{3}, \frac{2}{3}; z) = 3^{2/3} \mathrm{Ai}(-z/3^{1/3})$$

with the Airy function $\mathrm{Ai}(z)$. Finally, we rewrite the formulae (24), (25) by using the Kummer formula ([25, Chapter 6])

$${}_1F_1(a; c; z) = e^z\ {}_1F_1(c-a; c; -z)$$

in the form

$$\phi(-\frac{1}{2}, -n; z) = e^{-z^2/4} z P_n(z^2),\ n = 0, 1, 2, \ldots, \tag{30}$$

$$\phi(-\frac{1}{2}, \frac{1}{2} - n; z) = e^{-z^2/4} Q_n(z^2),\ n = 0, 1, 2, \ldots, \tag{31}$$

where $P_n(z),\ Q_n(z)$ are polynomials of degree n defined as

$$P_n(z) = \frac{(-1)^{n+1}}{\pi} \Gamma(3/2+n)\ {}_1F_1(-n; 3/2; \frac{z}{4}),$$

$$Q_n(z) = \frac{(-1)^n}{\pi} \Gamma(1/2+n)\ {}_1F_1(-n; 1/2; \frac{z}{4}).$$

2.3. Laplace transform pairs related to the Wright function

In the case $\rho > 0$ the Wright function is an entire function of order less than 1 and consequently its Laplace transform can be obtained by transforming term-by-term its Taylor expansion (1) in the origin. As a result we get ($0 \le t < +\infty$,

$s \in \mathbf{C},\ 0 < \epsilon < |s|,\ \epsilon$ arbitrarily small)

$$\begin{aligned} \phi(\rho,\beta;\pm t) \ &\div\ \mathcal{L}[\phi(\rho,\beta;\pm t);s] = \int_0^\infty e^{-st}\phi(\rho,\beta;\pm t)\,dt \qquad (32)\\ &= \int_0^\infty e^{-st}\sum_{k=0}^\infty \frac{(\pm t)^k}{k!\Gamma(\rho k+\beta)}\,dt = \sum_{k=0}^\infty \frac{(\pm 1)^k}{k!\Gamma(\rho k+\beta)}\int_0^\infty e^{-st}t^k\,dt\\ &= \frac{1}{s}\sum_{k=0}^\infty \frac{(\pm s^{-1})^k}{\Gamma(\rho k+\beta)} = \frac{1}{s}E_{\rho,\beta}(\pm s^{-1}),\quad \rho>0,\ \beta\in\mathbf{C}, \end{aligned}$$

where $\div$ denotes the juxtaposition of a function $\varphi(t)$ with its Laplace transform $\tilde{\varphi}(s)$, and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha>0,\ \beta\in\mathbf{C}, \qquad (33)$$

is the generalized Mittag-Leffler function. In this case the resulting Laplace transform turns out to be analytic, vanishing at infinity and exhibiting an essential singularity at $s=0$.

For $-1<\rho<0$ the just applied method cannot be used since then the Wright function is an entire function of order greater than one. The existence of the Laplace transform of the function $\phi(\rho,\beta;-t),\ t>0$, follows in this case from Theorem 2.1.3, which says us that the function $\phi(\rho,\beta;z)$ is exponentially small for large z in a sector of the plane containing the negative real semi-axis. To get the transform in this case we use the idea given in Mainardi [23]. Recalling the integral representation (2) we have $(-1<\rho<0)$

$$\begin{aligned} \phi(\rho,\beta;-t) \ &\div\ \int_0^\infty e^{-st}\phi(\rho,\beta;-t)\,dt = \int_0^\infty e^{-st}\frac{1}{2\pi i}\int_{\mathrm{Ha}} e^{\zeta-t\zeta^{-\rho}}\zeta^{-\beta}\,d\zeta\,dt\\ &= \frac{1}{2\pi i}\int_{\mathrm{Ha}} e^{\zeta}\zeta^{-\beta}\int_0^\infty e^{-t(s+\zeta^{-\rho})}\,dt\,d\zeta \qquad (34)\\ &= \frac{1}{2\pi i}\int_{\mathrm{Ha}} \frac{e^{\zeta}\zeta^{-\beta}}{s+\zeta^{-\rho}}\,d\zeta = E_{-\rho,\beta-\rho}(-s), \end{aligned}$$

again with the generalized Mittag-Leffler function according to (33). We use here the integral representation (see Djrbashian [2], Gorenflo and Mainardi [10])

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\mathrm{Ha}} \frac{e^{\zeta}\zeta^{\alpha-\beta}}{\zeta^\alpha - z}\,d\zeta, \qquad (35)$$

which is obtained by substituting the Hankel representation (3) for the reciprocal of the gamma function into the series representation (33).

The relation (34) was given in Djrbashian and Bagian [4] (see also Djrbashian [3]) in the case $\beta\geq 0$ as a representation of the generalized Mittag-Leffler function

in the whole complex plane as a Laplace integral of an entire function and without identifying this function as the known Wright function. They also gave (in slightly different notations) the more general representation

$$E_{\alpha_2,\beta_2}(z) = \int_0^\infty E_{\alpha_1,\beta_1}(zt^{\alpha_1})t^{\beta_1-1}\phi(-\alpha_2/\alpha_1, \beta_2 - \beta_1\frac{\alpha_2}{\alpha_1}; -t)\,dt,$$

$$0 < \alpha_2 < \alpha_1,\ \beta_1, \beta_2 > 0.$$

An important particular case of the Laplace transform pair (34) is given by

$$M(t;\beta) \div E_\beta(-s),\ 0 < \beta < 1, \tag{36}$$

where $M(t;\beta)$ is the Mainardi function given by (28) and

$$E_\alpha(z) = E_{\alpha,1}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)},\ \alpha > 0 \tag{37}$$

is the (standard) Mittag-Leffler function. The formula (36) contains, in particular, the well-known Laplace transform pair

$$M(t;1/2) = \frac{1}{\sqrt{\pi}}\exp(-t^2/4) \div E_{1/2}(-s) = \exp(s^2)\operatorname{erfc}(s),\ s \in \mathbf{C}.$$

Using the relation

$$\int_0^\infty t^n f(t)\,dt = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t);s],$$

the Laplace transform pair (34) and the series representation of the generalized Mittag-Leffler function (33) we can compute all the moments of the Wright function $\phi(\rho,\beta;-t),\ -1 < \rho < 0$ in $\mathbf{R}^+$:

$$\int_0^\infty t^n\phi(\rho,\beta;-t)\,dt = \frac{n!}{\Gamma(-\rho n + \beta - \rho)},\ n \in \mathbf{N}_0 = \{0,1,2,\ldots\}.$$

For the Mainardi function $M(t;\beta),\ 0 < \beta < 1$ we obtain from this formula the normalization property in $\mathbf{R}^+$ $(n = 0)$

$$\int_0^\infty M(t;\beta)\,dt = 1$$

and the moments in the form

$$\int_0^\infty t^n M(t;\beta)\,dt = \frac{n!}{\Gamma(\beta n + 1)},\ n \in \mathbf{N}.$$

Now we introduce the function (Mainardi [23])

$$F(z;\beta) = \phi(-\beta, 0; -z),\ 0 < \beta < 1, \tag{38}$$

which is connected with the function $M(z;\beta)$ by the relation

$$F(z;\beta) = \beta z M(z;\beta). \tag{39}$$

For this function we can prove the relation

$$\frac{1}{t}F(\lambda t^{-\beta};\beta) = \frac{\beta\lambda}{t^{\beta+1}}M(\lambda t^{-\beta};\beta) \div \exp(-\lambda s^{\beta}),\ 0<\beta<1,\ \lambda>0. \tag{40}$$

Indeed, following Mainardi [23] and using the integral representation (2) we get

$$\begin{aligned}\mathcal{L}^{-1}[\exp(-\lambda s^{\beta});t] &= \frac{1}{2\pi i}\int_{\mathrm{Ha}} e^{st-\lambda s^{\beta}}ds = \frac{1}{2\pi i\, t}\int_{\mathrm{Ha}} e^{\zeta-\lambda t^{-\beta}\zeta^{\beta}}d\zeta \\ &= \frac{1}{t}F(\lambda t^{-\beta};\beta) = \frac{\beta\lambda}{t^{\beta+1}}M(\lambda t^{-\beta};\beta).\end{aligned} \tag{41}$$

The Laplace transform pair (40) was formerly given by Pollard [33] and by Mikusiński [27].

By applying the formula for differentiation of the image of the Laplace transform to (40) we get the Laplace transform pair useful for our further discussions:

$$\frac{1}{t^{\beta}}M(\lambda t^{-\beta};\beta) \div s^{\beta-1}\exp(-\lambda s^{\beta}),\ 0<\beta<1,\ \lambda>0. \tag{42}$$

In the general case, using the same method as in (41), we get (see Stanković [39]) the Laplace transform pair

$$t^{\beta-1}\phi(\rho,\beta;-\lambda t^{\rho}) \div s^{-\beta}\exp(-\lambda s^{-\rho}),\ -1<\rho<0,\ \lambda>0.$$

Stanković [39] also gave some other Laplace transform pairs related to the Wright function including

$$t^{\frac{\beta}{2}-1}\phi(\rho,\beta;-t^{\frac{\rho}{2}}) \div \frac{\sqrt{\pi}}{2^{\beta}}s^{-\frac{\beta}{2}}\phi(\frac{\rho}{2},\frac{\beta+1}{2};-2^{-\rho}s^{-\frac{\rho}{2}}),\quad -1<\rho<0,$$

$$t^{-\beta}\exp(-t^{-\rho}\cos(\rho\pi))\sin(\beta\pi - t^{-\rho}\sin(\rho\pi)) \div \pi s^{\beta-1}\phi(\rho,\beta;-s^{\rho}), -1<\rho<0,\ \beta<1.$$

2.4. The Wright function as an entire function of completely regular growth

The fact that the function (1) is an entire function for all values of the parameters $\rho > -1$ and $\beta \in \mathbf{C}$ was already known to Wright (Wright [42], [44]). In the paper Djrbashian and Bagian [4] (see also Djrbashian [3]) the order and type of this function as well as an estimate of its indicator function were given for the case $-1 < \rho < 0$. Wright [44] also remarked that the zeros of the function (1) lie near the positive real semi-axis if $-1/3 \le \rho < 0$ and near the two lines $\arg z = \pm\frac{1}{2}\pi(3\rho+1)$ if $-1 < \rho < -1/3$. In this paper we continue the investigations of the Wright function from the viewpoint of the theory of entire functions.

We give exact formulae for the order, the type and the indicator function of the entire function $\phi(\rho,\beta;z)$ for $\rho > -1,\ \beta \in \mathbf{C}$. On the basis of these results the problem of distribution of zeros of the Wright function is considered. In all cases this function is shown to be a function of completely regular growth.

The order and the type of the Wright function are obtained in a straightforward way by using the standard formulae for the order p and the type σ of an entire function $f(z)$ defined by the power series $f(z) = \sum_{k=0}^{\infty} c_n z^n$

$$p = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)},\ (\sigma e p)^{1/p} = \limsup_{n\to\infty} n^{1/p}\sqrt[n]{c_n}$$

and the Stirling asymptotic formula

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z}[1 + O(1/z)],\ |\arg z| \le \pi - \epsilon,\ \epsilon > 0,\ |z| \to \infty.$$

We thus obtain the following result.

THEOREM 2.4.1. *The Wright function $\phi(\rho,\beta;z)$, $\rho > -1$, $\beta \in \mathbf{C}$ ($\beta \ne -n$, $n = 0,1,\ldots$ if $\rho = 0$) is an entire function of finite order p and the type σ given by*

$$p = \frac{1}{1+\rho},\ \sigma = (1+\rho)|\rho|^{-\frac{\rho}{1+\rho}}. \tag{43}$$

REMARK 2.4.1. In the case $\rho = 0$ the Wright function is reduced to the exponential function with the constant factor $1/\Gamma(\beta)$, which turns out to vanish identically for $\beta = -n,\ n = 0,1,\ldots$. For all other values of the parameter β and $\rho = 0$ the formulae (43) (with $\sigma = \lim\limits_{\rho\to 0}(1+\rho)|\rho|^{-\frac{\rho}{1+\rho}} = 1$) are still valid.

The basic characteristic of the growth of an entire function $f(z)$ of finite order p in different directions is its indicator function $h(\theta)$, $|\theta| \le \pi$ defined by the equation

$$h(\theta) = \limsup_{r\to+\infty} \frac{\log|f(re^{i\theta})|}{r^p}. \tag{44}$$

To find the indicator function $h_\rho(\theta)$ of the entire function $\phi(\rho,\beta;z)$ of finite order p given by (43) its asymptotics given in Section 2.1 are used. By direct evaluations we arrive at the following theorem.

THEOREM 2.4.2. *Let $\rho > -1$, $\beta \in \mathbf{C}$ ($\beta \ne -n$, $n = 0,1,\ldots$ if $\rho = 0$). Then the indicator function $h_\rho(\theta)$ of the Wright function $\phi(\rho,\beta;z)$ is given by the formulae*

$$h_\rho(\theta) = \sigma\cos p\theta,\ |\theta| \le \pi \tag{45}$$

in the case $\rho \geq 0$,

$$h_\rho(\theta) = \begin{cases} -\sigma \cos p(\pi + \theta), & -\pi \leq \theta \leq 0, \\ -\sigma \cos p(\theta - \pi), & 0 \leq \theta \leq \pi \end{cases} \tag{46}$$

in the cases (a) $-1/3 \leq \rho < 0$, *(b)* $\rho = -1/2,\ \beta = -n,\ n = 0, 1, \ldots$ *and (c)* $\rho = -1/2,\ \beta = 1/2 - n,\ n = 0, 1, \ldots$,

$$h_\rho(\theta) = \begin{cases} -\sigma \cos p(\pi + \theta), & -\pi \leq \theta \leq \frac{3}{2}\frac{\pi}{p} - \pi, \\ 0, & |\theta| \leq \pi - \frac{3}{2}\frac{\pi}{p}, \\ -\sigma \cos p(\theta - \pi), & \pi - \frac{3}{2}\frac{\pi}{p} \leq \theta \leq \pi \end{cases} \tag{47}$$

in the case $-1 < \rho < -1/3$ ($\beta \neq -n,\ n = 0, 1, \ldots$ *and* $\beta \neq 1/2 - n,\ n = 0, 1, \ldots$ *if* $\rho = -1/2$), *where* p *and* σ *are the order and type of the Wright function, respectively, given by* (43).

Remark 2.4.2. It can be seen from the formulae (45), (46) that the indicator function $h_\rho(\theta)$ of the Wright function $\phi(\rho, \beta; z)$ is reduced to the function $\cos\theta$ – the indicator function of the exponential function e^z – if $\rho \to 0$. This property is not valid for another generalization of the exponential function – the Mittag-Leffler function (37). Even though

$$E_1(z) = e^z,$$

the indicator function of the Mittag-Leffler function given for $0 < \alpha < 2,\ \alpha \neq 1$ by ([6, Chapter 2.7])

$$h(\theta) = \begin{cases} \cos\theta/\alpha, & |\theta| \leq \frac{\pi\alpha}{2}, \\ 0, & \frac{\pi\alpha}{2} \leq |\theta| \leq \pi \end{cases}$$

does not coincide with the indicator function of e^z if $\alpha \to 1$.

We consider now the problem of distribution of zeros of the Wright function in the case $\rho > -1,\ \beta \in \mathbf{R}$. To get the asymptotics of zeros of the Wright function we use its asymptotic expansions (4), (6), (13), (15), (18), (19) and the method applied by M.M. Djrbashian in [2, Chapter 1.2] to solve the problem of distribution of zeros of the generalized Mittag-Leffler function $E_{\rho,\mu}(z)$. This method consists in finding the asymptotics of zeros of the main terms of the asymptotic expansions, applying the Rouché theorem to show that the function under consideration and the main terms of its asymptotic expansions have the same number of zeros inside of specially chosen contours and after that in estimation of the diameter of the domains bounded by the contours. The proofs of the results given below are straightforward but have many technical details and are omitted in this paper. It turns out, that in dependence of the value of the parameter $\rho > -1$ and the real parameter β, there are five different situations:

1) for $\rho > 0$ all zeros with large enough absolute values are simple and are lying on the negative real semi-axis;

2) in the case $\rho = 0$ the Wright function becomes the exponential function with a constant factor (equal to zero if $\beta = -n,\ n = 0, 1, \ldots$) and it has no zeros;

3) for $-1/3 \leq \rho < 0$ all zeros with large enough absolute values are simple and are lying on the positive real semi-axis;

4) in the cases $\rho = -1/2,\ \beta = -n,\ n = 0, 1, \ldots$ and $\rho = -1/2,\ \beta = 1/2 - n,\ n = 0, 1, \ldots$ the Wright function has exactly $2n + 1$ and $2n$ zeros, respectively;

5) for $-1 < \rho < -1/3$ (excluding the case 4)) all zeros with large enough absolute values are simple and are lying in the neighbourhoods of the rays $\arg z = \pm\frac{1}{2}\pi(-1 - 3\rho)$.

We now give the precise results.

THEOREM 2.4.3. *Let $\{\gamma_k\}_1^\infty$ be the sequence of zeros of the function $\phi(\rho, \beta; z)$, $\rho \geq -1/3,\ \rho \neq 0,\ \beta \in \mathbf{R}$, where $|\gamma_k| \leq |\gamma_{k+1}|$ and each zero is counted according to its multiplicity. Then:*

A. *In the case $\rho > 0$ all zeros with large enough k are simple and are lying on the negative real semi-axis. The asymptotic formula*

$$\gamma_k = -\left(\frac{\pi k + \pi(p\beta - \frac{p-1}{2})}{\sigma \sin \pi p}\right)^{\frac{1}{p}} \left\{1 + O(k^{-2})\right\},\ k \to +\infty \tag{48}$$

is true. Here and in the next formulae p and σ are the order and type of the Wright function given by (43)*, respectively.*

B. *In the case $-1/3 \leq \rho < 0$ all zeros with large enough k are simple, lying on the positive real semi-axis and the asymptotic formula*

$$\gamma_k = \left(\frac{\pi k + \pi(p\beta - \frac{p-1}{2})}{-\sigma \sin \pi p}\right)^{\frac{1}{p}} \left\{1 + O(k^{-2})\right\},\ k \to +\infty \tag{49}$$

is true.

REMARK 2.4.3. Combining the representation (20) with the asymptotic formula (48) we get the known formula (see, for example [40, p.506]) for asymptotic expansion of the large zeros r_k of the Bessel function $J_\nu(z)$:

$$r_k = \pi(k + \frac{1}{2}\nu - \frac{1}{4}) + O(k^{-1}),\ k \to \infty.$$

REMARK 2.4.4. In the cases $\rho = -1/2,\ \beta = -n,\ n = 0, 1, \ldots$ and $\rho = -1/2,\ \beta = 1/2 - n,\ n = 0, 1, \ldots$ the Wright function can be represented by the formulae (30), (31) and, consequently, has exactly $2n + 1$ and $2n$ zeros in the complex plane, respectively.

It follows from the asymptotic formulae (11), (16) and (19) that all zeros of the function $\phi(\rho,\beta;z)$ in the case $-1<\rho<-1/3$ with large enough absolute values are lying inside of the angular domains

$$\Omega_{\epsilon}^{(\pm)}=\left\{z:\ \left|\arg z\mp\left(\pi-\frac{3\pi}{2p}\right)\right|<\epsilon\right\},$$

where ϵ is any number of the interval $(0,\min\{\pi-\frac{3\pi}{2p},\frac{3\pi}{2p}\})$. Consequently, the function $\phi(\rho,\beta;z)$ has on the real axis only finitely many zeros. Let

$$\{\gamma_k^{(+)}\}_1^{\infty}\in G^{(+)}=\{z:\ \Im>0\},\ \{\gamma_k^{(-)}\}_1^{\infty}\in G^{(-)}=\{z:\ \Im<0\}$$

be sequences of zeros of the function $\phi(\rho,\beta;z)$ in the upper and lower half-plane, respectively, such that $|\gamma_k^{(+)}|\le|\gamma_{k+1}^{(+)}|$, $|\gamma_k^{(-)}|\le|\gamma_{k+1}^{(-)}|$, and each zero is counted according to its multiplicity.

THEOREM 2.4.4. *In the case $-1<\rho<-1/3$ ($\beta\neq-n,\ n=0,1,\ldots$ and $\beta\neq1/2-n,\ n=0,1,\ldots$ if $\rho=-1/2$) all zeros of the function $\phi(\rho,\beta;z),\ \beta\in\mathbf{R}$ with large enough k are simple and the asymptotic formula*

$$\gamma_k^{(\pm)}=e^{\pm i(\pi-\frac{3\pi}{2p})}\left(\frac{2\pi k}{\sigma}\right)^{\frac{1}{p}}\left\{1+O\left(\frac{\log k}{k}\right)\right\},\ k\to+\infty \tag{50}$$

is true.

Summarizing all results concerning the asymptotic behaviour of the Wright function, its indicator function and the distribution of its zeros, we get the theorem.

THEOREM 2.4.5. *The Wright function $\phi(\rho,\beta;z),\ \rho>-1$ is an entire function of completely regular growth.*

We recall ([18, Chapter 3]) that an entire function $f(z)$ of finite order p is called a function of completely regular growth (CRG-function) if for all $\theta,\ |\theta|\le\pi$, there exist a set $E_\theta\subset\mathbf{R}_+$ and the limit

$$\lim_{\substack{r\to+\infty\\ r\in E_\theta^*}}\frac{\log\mid f(re^{i\theta})\mid}{r^p}, \tag{51}$$

where

$$E_\theta^*=\mathbf{R}_+\setminus E_\theta,\quad \lim_{r\to+\infty}\frac{\mathrm{mes}E_\theta\bigcap(0,r)}{r}=0.$$

It is known ([6, Chapter 2.6]) that zeros of a CRG-function $f(z)$ are regularly distributed, namely, they possess the finite angular density

$$\lim_{r\to+\infty}\frac{n(r,\theta)}{r^p}=\nu(\theta), \tag{52}$$

where $n(r,\theta)$ is the number of zeros of $f(z)$ in the sector $0 < \arg z < \theta,\ |z| < r$ and p is the order of $f(z)$. ¿From the other side, the angular density $\nu(\theta)$ is connected with the indicator function $h(\theta)$ of a CRG-function. In particular (see [6, Chapter 2.6]), the jump of $h'(\theta)$ at $\theta = \theta_0$ is equal to $2\pi p\Delta$, where Δ is the density of zeros of $f(z)$ in an arbitrarily small angle containing the ray $\arg z = \theta_0$.

In our case we get from Theorem 2.4.2, that the derivative of the indicator function of the Wright function has the jump $2\sigma p \sin \pi p$ at $\theta = \pi$ for $\rho > 0$, the same jump at $\theta = 0$ for $-1/3 < \rho < 0$, and the jump σp at $\theta = \pm(\pi - \frac{3\pi}{2p})$ for $-1 < \rho < -1/3$ ($\beta \neq -n,\ n = 0,1,\ldots$ and $\beta \neq 1/2 - n,\ n = 0,1,\ldots$ if $\rho = -1/2$), where again p and σ are the order and type of the Wright function, respectively, given by (43); if $\rho = 0$ or $\rho = -1/2$ and either $\beta = -n,\ n = 0,1,\ldots$, or $\beta = 1/2 - n,\ n = 0,1,\ldots$, the derivative of the indicator function has no jumps. As we see, the behaviour of the derivative of the indicator function of the Wright function is in accordance with the distribution of its zeros given by Theorems 2.4.3, 2.4.4 and Remark 2.4.4 as predicted by the general theory of the CRG-functions.

3. Some applications of the Wright function

3.1. Asymptotic theory of partitions

Historically the first application of the Wright function was connected with the asymptotic theory of partitions. Extending the results of Hardy and Ramanujan about asymptotic expansion of the function $p(n),\ n \in \mathbf{N}$, the number of partitions of n, Wright [41] considered the more general problem, namely, to find an asymptotic expansion for the function $p_k(n),\ n \in \mathbf{N}$, the number of partitions of n into perfect k-th powers. Following Hardy and Ramanujan, Wright considered the generating function for the sequence $\{p_k(1),\ p_k(2),\ldots\}$ which is given by

$$f_k(z) = \prod_{l=1}^{\infty}(1 - z^{l^k})^{-1} = 1 + \sum_{n=1}^{\infty} p_k(n) z^n,\ |z| < 1.$$

Then

$$p_k(n) = \frac{1}{2\pi i}\int_{\mathrm{C}} \frac{f_k(z)\,dz}{z^{n+1}},$$

the contour C being the periphery of the circle with center in the point $z = 0$ and radius $r = 1 - \frac{1}{n}$. Let the contour be divided into a large number of small arcs, each associated with a point

$$\alpha_{p,q} = \exp(2p\pi i/q),\ p, q \in \mathbf{N}.$$

Taking the arc associated with $\alpha_{0,1} = 1$ as typical, it can be shown that on this arc the generating function $f_k(z)$ has the representation

$$f_k(z) \sim \frac{z^j}{(2\pi)^{\frac{1}{2}(k+1)}} \left(\log\frac{1}{z}\right)^{\frac{1}{2}} \exp\left(\frac{\Gamma(1 + (1/k))\zeta(1 + (1/k))}{(\log(1/z))^{1/k}}\right),\ z \to 1, \qquad (53)$$

where j is a real number depending on k and $\zeta(z)$ is the Riemann zeta-function. Then, on this arc, $f_k(z)$ is approximated to by an auxiliary function $F_k(z)$, which has a singularity at $z = 1$ of the type of the right-hand side of (53). If the z-plane is cut along the interval $(1, \infty)$ of the real axis, $F_k(z)$ is regular and one-valued for all values of z except those on the cut. The power series for $F_k(z)$ has coefficients given in terms of the entire function $\phi(\rho, \beta; z)$ and, by using this power series, an asymptotic expansion can be found for $p_k(n)$.

In the paper [41] Wright gave some properties of the function $\phi(\rho, \beta; z)$ in the case $\rho > 0$, including its asymptotics and integral representation (2). He proved on this base the following two theorems.

THEOREM 3.1.1. *Let* $\alpha,\ \beta,\ \gamma \in \mathbf{C},\ \alpha \neq 0,\ \rho > 0,\ m \in \mathbf{N},\ m > \Re(\gamma)$,

$$F(z) = F(\rho, \alpha, \beta, \gamma; z) := \sum_{n=m}^{\infty} (n - \gamma)^{\beta - 1} \phi(\rho, \beta; \alpha(n - \gamma)^{\rho}) z^n. \tag{54}$$

If a cut is made in the z-plane along the segment $(1, \infty)$ of the real axis, then $F(z)$ is regular and one-valued in the interior of the region thus defined.

THEOREM 3.1.2. *Let*

$$G(z) = F(z) - \chi(z),$$

where $F(z)$ is defined by (54) *and*

$$\chi(z) = \frac{z^{\gamma}}{(\log(1/z))^{\beta}} \exp\left(\frac{\alpha}{(\log(1/z))^{\rho}}\right).$$

If a cut is made in the z-plane along the segment $(-\infty, 0)$ of the real axis, then $G(z)$ is regular and one-valued in the interiour of the region thus defined.

We see that the function $F(x)$ has a singularity of the type of $\chi(z)$ at $z = 1$. In the case of the function $F_k(z)$ used to get an asymptotic expansion for the function $p_k(n)$ the values

$$\rho = \frac{1}{k},\ \alpha = \Gamma(1 + \frac{1}{k})\zeta(1 + \frac{1}{k}),\ \beta = -\frac{1}{2},\ \gamma = \frac{1}{24}$$

should be taken in the previous two theorems.

3.2. Fractional diffusion-wave equation

Another field in which the Wright function plays a very important role is that of partial differential equations of fractional order. Following Gorenflo, Mainardi and Srivastava [14] and Mainardi [20]–[23] we consider the fractional diffusion-wave equation which is obtained from the classical diffusion or wave equation by

replacing the first- or second-order time derivative by a fractional derivative of order α with $0 < \alpha \leq 2$:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathcal{D}\frac{\partial^2 u(x,t)}{\partial x^2}, \quad \mathcal{D} > 0, \quad 0 < \alpha \leq 2. \tag{55}$$

Here the field variable $u = u(x,t)$ is assumed to be a causal function of time, i.e. vanishing for $t < 0$, and the fractional derivative is taken in the Caputo sense:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in \mathbf{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n}\, d\tau, & n-1 < \alpha < n. \end{cases} \tag{56}$$

We refer to the equation (55) as to the *fractional diffusion* and to the *fractional wave* equation in the cases $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$, respectively. The difference between these two cases can be seen in the formula for the Laplace transform of the Caputo fractional derivative of order α (see Mainardi [23]):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \div s^\alpha \tilde{u}(x,s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} \frac{\partial^k u(x,t)}{\partial t^k}|_{t=0+}, \; n-1 < \alpha \leq n, \; n \in \mathbf{N}. \tag{57}$$

Extending the conventional analysis to the equation (55), and denoting by $g(x)$ and $h(x)$ two given, sufficiently well-behaved functions, the basic boundary-value problems can be formulated as follows ($0 < \alpha \leq 1$):

a) Cauchy problem

$$u(x,0+) = g(x), \; -\infty < x < +\infty; \quad u(\mp\infty, t) = 0, \; t > 0; \tag{58}$$

b) Signalling problem

$$u(x,0+) = 0, \; x > 0; \quad u(0+,t) = h(t), \; u(+\infty,t) = 0, \; t > 0. \tag{59}$$

If $1 < \alpha \leq 2$ the initial values of the first time-derivative of the field variable, $\dot{u}(x,0+)$, should be added to to the conditions (58) and (59). To ensure the continuous dependence of the solutions on the parameter α in the transition from $\alpha = 1-$ to $\alpha = 1+$, we agree to assume $\dot{u}(x,0+) = 0$.

Since these problems are well studied in the cases $\alpha = 1$ and $\alpha = 2$ we restrict ourselves in the further considerations to the case $0 < \alpha < 2, \; \alpha \neq 1$. For the sake of convenience we use the abbreviation

$$\beta = \frac{\alpha}{2}, \tag{60}$$

which implies $0 < \beta < 1$.

Let us introduce the Green functions $\mathcal{G}_c(x,t;\beta)$ and $\mathcal{G}_s(x,t;\beta)$ for the Cauchy and signalling problems for the equation (55), respectively, which represent the fundamental solutions of these problems (with $g(x)=\delta(x)$ in (58) and $h(t)=\delta(t)$ in (59)). Using the Green functions, the solutions of the two basic problems can be given, respectively, by

$$u(x,t;\beta)=\int_{-\infty}^{+\infty}\mathcal{G}_c(x-\xi,t;\beta)g(\xi)\,d\xi, \tag{61}$$

$$u(x,t;\beta)=\int_{0}^{t}\mathcal{G}_s(x,t-\tau;\beta)h(\tau)\,d\tau. \tag{62}$$

To get the Green functions $\mathcal{G}_c(x,t;\beta)$ and $\mathcal{G}_s(x,t;\beta)$ the technique of the Laplace transform is used. We consider at first the Cauchy problem (58) for the equation (55) with $g(x)=\mathcal{G}_c(x,0+;\beta)=\delta(x)$ (and $\dot{\mathcal{G}}_c(x,0+;\beta)=0$ if $1/2<\beta<1$). Denoting the Laplace transform of the Green function by $\tilde{\mathcal{G}}_c(x,s;\beta)$ and using the formula (57) we arrive after application of the Laplace transform to the Cauchy problem {(55), (58)} to the non-homogeneous differential equation

$$\mathcal{D}\frac{d^2\tilde{\mathcal{G}}_c}{dx^2}-s^{2\beta}\tilde{\mathcal{G}}_c=-\delta(x)s^{2\beta-1},\ -\infty<x<+\infty \tag{63}$$

with the boundary conditions

$$\tilde{\mathcal{G}}_c(\mp\infty,s;\beta)=0. \tag{64}$$

The problem {(63), (64)} has a solution (see, for example, Mainardi [23])

$$\tilde{\mathcal{G}}_c(x,s;\beta)=\frac{1}{2\sqrt{\mathcal{D}}\,s^{1-\beta}}e^{-(|x|/\sqrt{\mathcal{D}})s^{\beta}},\ -\infty<x<+\infty. \tag{65}$$

Comparing this relation with the Laplace transform pair (42) we represent the Green function for the Cauchy problem {(55), (58)} in the form

$$\mathcal{G}_c(x,t;\beta)=\frac{r}{2\sqrt{\mathcal{D}}|x|}M(r/\sqrt{\mathcal{D}};\beta),\ t>0, \tag{66}$$

where

$$r=|x|\,t^{-\beta}$$

is the *similarity variable* and $M(z;\beta)$ is the Mainardi function (28) given in terms of the Wright function.

For the signalling problem {(55), (59)} (with $h(t)=\delta(t)$) the application of the Laplace transform leads to the homogeneous differential equation

$$\mathcal{D}\frac{d^2\tilde{\mathcal{G}}_s}{dx^2}-s^{2\beta}\tilde{\mathcal{G}}_s=0,\ x\geq 0 \tag{67}$$

with the boundary conditions

$$\tilde{\mathcal{G}}_s(0+,s;\beta)=1,\ \tilde{\mathcal{G}}_s(+\infty,s;\beta)=0. \tag{68}$$

Solving this equation, we obtain

$$\tilde{\mathcal{G}}_s(x,s;\beta) = e^{-(x/\sqrt{D})s^\beta},\ x \geq 0. \tag{69}$$

Using the Laplace transform pair (40) we get the Green function $\mathcal{G}_s(x,t;\beta)$ for the signalling problem {(55), (59)} in the form

$$\mathcal{G}_s(x,t;\beta) = \frac{\beta r}{\sqrt{\mathcal{D}}t} M(r/\sqrt{\mathcal{D}};\beta),\ t > 0,\ x \geq 0, \tag{70}$$

where

$$r = x\, t^{-\beta} \tag{71}$$

is the *similarity variable* and $M(z;\beta)$ is the Mainardi function (28).

For more results in FPDE we refer, for example, to Engler [5], Fujita [7], Gorenflo and Mainardi [11], [12], Mainardi [20]-[23], Podlubny [32], Prüss [34], Saichev and Zaslavsky [35], Samko et. al. [36], Schneider and Wyss [37] and by Wyss [45].

Some applications of FPDE have been considered in papers by several authors including Giona and Roman [9], Hilfer [15], Mainardi [21], Metzler et al. [26], Nigmatullin [28], Pipkin [31], Podlubny [32].

3.3. Scale invariant solutions of FPDE

Let us consider the abstract equation

$$F(u) \ = \ 0, \quad u = u(x,t). \tag{72}$$

First we give some definitions concerning the similarity method.

Definition 3.3.1. A one-parameter family of scaling transformations, denoted by T_λ, is a transformation of (x,t,u)-space of the form

$$\bar{x} \ = \ \lambda^a x, \quad \bar{t} \ = \ \lambda^b t, \quad \bar{u} \ = \ \lambda^c u, \tag{73}$$

where a, b, and c are constants and λ is a real parameter restricted to an open interval I containing $\lambda = 1$.

Definition 3.3.2. The equation (72) is invariant under the one-parameter family T_λ of scaling transformations (73) iff T_λ takes any solution u of (72) to a solution $\bar{u}$ of the same equation:

$$\bar{u} \ = \ T_\lambda u \quad \text{and} \quad F(\bar{u}) \ = \ 0. \tag{74}$$

Definition 3.3.3. A real-valued function $\eta(x,t,u)$ is called an *invariant* of the one-parameter family T_λ, if it is unaffected by the transformations, in other words:

$$\eta(T_\lambda(x,t,u)) \ = \ \eta(x,t,u) \quad \text{for all} \quad \lambda \in I.$$

On the half-space $\{(x,t,u) : x > 0,\ t > 0\}$, the invariants of the family of scaling transformations (73) are provided by the functions (see [29])

$$\eta_1(x,t,u) = xt^{-a/b},\ \eta_2(x,t,u) = t^{-c/b}u. \tag{75}$$

If the equation (72) is a second order partial differential equation of the form

$$G(x,\ t,\ u,\ u_x,\ u_t,\ u_{xx},\ u_{tt},\ u_{xt}) \ = \ 0, \tag{76}$$

and this equation is invariant under T_λ, given by (73), then the transformation

$$u(x,t) \ = \ t^{c/b}v(z), \quad z \ = \ xt^{-a/b} \tag{77}$$

reduces the equation (76) to a second order ordinary differential equation of the form

$$g(z,\ v,\ v',\ v'') \ = \ 0. \tag{78}$$

For a proof of this fact we refer in the case of general Lie group methods to [29]. In some cases it can be easily checked directly.

Recently, the scale-invariant solutions for the equation (55) (with the fractional derivative in the Caputo and Riemann-Liouville sense) and for the more general time- and space-fractional partial differential equation (with the Riemann-Liouville space-fractional derivative of order $\beta \leq 2$ instead of the second order space derivative in the equation (55)) have been obtained by Gorenflo, Luchko and Mainardi [13], Buckwar and Luchko [1] and Luchko and Gorenflo [19], respectively. In all cases these solutions have been given in terms of the Wright and the generalized Wright functions. Here we present some results from these papers.

At first we determine a group of scaling transformations for the fractional diffusion-wave equation (55) on the semi-axis ($x \geq 0$) with the Caputo fractional derivative given by (56). We have in this case the following theorem.

THEOREM 3.3.1. *Let T_λ be a one parameter group of scaling transformations for the equation (55) of the form $T_\lambda \circ (x,t,u) = (\lambda x,\ \lambda^b t,\ \lambda^c u)$. Then,*

$$b = \frac{2}{\alpha} \tag{79}$$

and the invariants of this group T_λ are given by the expressions

$$\eta_1(x,t) = xt^{-1/b} = xt^{-\alpha/2},\ \eta_2(x,t,u) = t^{-c/b}u = t^{-\gamma}u \tag{80}$$

with a real parameter $\gamma = c\alpha/2$.

REMARK 3.3.1. We note that the first scale-invariant η_1 of *(80)* coincides with the similarity variable *(71)* which was used to define the Green function of the signalling boundary-value problem for the equation *(55)*. It is a consequence of the fact that the equation *(55)* is invariant under the corresponding group of scaling transformations.

It follows from the general theory of Lie groups and the previous theorem that the scale-invariant solutions of the equation (55) should have the form

$$u(x,t) = t^{\gamma} v(y),\ y = xt^{-\alpha/2}. \tag{81}$$

Furthermore, the general theory says that the substitution (81) reduces the partial integro-differential equation (55) into an ordinary integro-differential equation with the unknown function $v(y)$.

THEOREM 3.3.2. *The reduced equation for the scale-invariant solutions of the equation* (55) *of the form* (81) *is given by*

$$({}_*P^{\gamma-n+1,\alpha}_{2/\alpha} v)(y) = \mathcal{D}v''(y),\ y > 0, \tag{82}$$

where the operator in the left-hand side is the Caputo type modification of the left-hand sided Erdélyi-Kober fractional differential operator defined for $0 < \delta,\ n-1 < \alpha \le n \in \mathbf{N}$ *by*

$$({}_*P^{\tau,\alpha}_{\delta} g)(y) := (K^{\tau,n-\alpha}_{\delta} \prod_{j=0}^{n-1} (\tau + j - \frac{1}{\delta} u \frac{d}{du}) g)(y),\ y > 0. \tag{83}$$

Here

$$(K^{\tau,\alpha}_{\delta} g)(y) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (u-1)^{\alpha-1} u^{-(\tau+\alpha)} g(yu^{1/\delta})\, du, & \alpha > 0, \\ g(y), & \alpha = 0 \end{cases} \tag{84}$$

is the left-hand sided Erdélyi-Kober fractional integral operator.

REMARK 3.3.2. As it follows from the definitions of the Caputo type modification of the Erdélyi-Kober fractional differential operator (83) and the Erdélyi-Kober fractional integral operator (84) in the case $\alpha = n \in \mathbf{N}$, the equation (82) for the scale-invariant solutions is a linear ordinary differential equation of order $\max\{n, 2\}$. In the case $\alpha = 1$ (the diffusion equation) we have

$$({}_*P^{\gamma,1}_{2} v)(y) = (\gamma - \frac{1}{2} y \frac{d}{dy}) v(y)$$

and (82) takes the form

$$\mathcal{D}v''(z) + \frac{1}{2} y v'(y) - \gamma v(y) = 0. \tag{85}$$

In the case $\alpha = 2$ (the wave equation) we get

$$
\begin{aligned}
({}_*P_1^{\gamma-1,2}v)(y) &= (\gamma - 1 - y\frac{d}{dy})(\gamma - y\frac{d}{dy})v(y) \\
&= y^2v''(y) - 2(\gamma-1)yv'(y) + \gamma(\gamma-1)v(y)
\end{aligned}
$$

and (82) is reduced to the ordinary differential equation of the second order:

$$
(y^2 - \mathcal{D})v''(y) - 2(\gamma-1)yv'(y) + \gamma(\gamma-1)v(y) = 0. \tag{86}
$$

The complete discussion of these cases one can find, for example, in [29]. The case $\alpha = n \in \mathbf{N},\ n > 2$ was considered in [1].

Solving the equation (83) we get the following theorems.

THEOREM 3.3.3. *The scale-invariant solutions of the fractional diffusion equation* (55) $(0 < \alpha \leq 1)$ *have the form*

$$
u(x,t) = C_1 t^{\gamma}\phi(-\frac{\alpha}{2}, 1+\gamma; -\frac{y}{\sqrt{\mathcal{D}}}) \tag{87}
$$

in the case $-1 < \gamma,\ \gamma \neq 0$, *and*

$$
u(x,t) = C_1\phi(-\frac{\alpha}{2}, 1; -\frac{y}{\sqrt{\mathcal{D}}}) + C_2 \tag{88}
$$

in the case $\gamma = 0$, *where* $y = xt^{-\frac{\alpha}{2}}$ *is the first scale invariant* (80) *and* C_1, C_2 *are arbitrary constants.*

THEOREM 3.3.4. *The scale-invariant solutions of the fractional wave equation* (55) $(1 < \alpha < 2)$ *have the form*

$$
\begin{aligned}
u(x,t) &= C_1 t^{\gamma}\phi(-\frac{\alpha}{2}, 1+\gamma; -\frac{y}{\sqrt{\mathcal{D}}}) \\
&+ C_2 t^{\gamma}\left(\frac{\mathcal{D}^{\frac{\gamma-1}{\alpha}}}{2}\phi(-\frac{\alpha}{2}, 1+\gamma; \frac{y}{\sqrt{\mathcal{D}}})\right. \\
&- \left.\frac{y^{2+2\frac{\gamma-1}{\alpha}}}{\mathcal{D}}\phi((-\alpha, 2-\alpha), (2, 3+2\frac{\gamma-1}{\alpha}); \frac{y^2}{\mathcal{D}})\right),
\end{aligned} \tag{89}
$$

in the case $1 - \alpha < \gamma < 1,\ \gamma \neq 1 - \frac{\alpha}{2},\ \gamma \neq 0$, *and*

$$
u(x,t) = C_1\phi(-\frac{\alpha}{2}, 1; -\frac{y}{\sqrt{\mathcal{D}}}) \tag{90}
$$

$$
+C_2\left(\frac{\mathcal{D}^{-\frac{1}{\alpha}}}{2}\phi(-\frac{\alpha}{2}, 1; \frac{y}{\sqrt{\mathcal{D}}}) - \frac{y^{2-\frac{2}{\alpha}}}{\mathcal{D}}\phi((-\alpha, 2-\alpha), (2, 3-\frac{2}{\alpha}); \frac{y^2}{\mathcal{D}})\right) + C_3
$$

in the case $\gamma = 0$, *where* $y = xt^{-\frac{\alpha}{2}}$ *is the first scale invariant* (80), $\phi((\mu, a), (\nu, b); z)$ *is the generalized Wright function given by* ($\mu + \nu > 0$)

$$\phi((\mu, a), (\nu, b); z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a + \mu k)\Gamma(b + \nu k)}, \qquad \mu, \nu \in \mathbf{R},\ a, b \in \mathbf{C}, \tag{91}$$

and C_1, C_2, C_3 *are arbitrary constants.*

For the elements of the theory of the generalized Wright function (91) including its integral representations and asymptotics we refer to Wright [43] in the case $\mu, \nu > 0$ and to Luchko and Gorenflo [19] in the case of one of the parameters $\mu,\ \nu$ being negative.

We consider now the equation (55) on the semi-axis $x \geq 0$ with the fractional derivative in the Riemann-Liouville sense:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in \mathbf{N}\ , \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} u(x,\tau)\, d\tau, & n-1 < \alpha < n. \end{cases} \tag{92}$$

Also in this case the scale-invariants of a one parameter group T_λ of scaling transformations for the equation (55) of the form $T_\lambda \circ (x, t, u) = (\lambda x,\ \lambda^b t,\ \lambda^c u)$ are given by Theorem 3.3.1.

Following Buckwar and Luchko [1] we restrict ourselves in the further discussion to the case of the group T_λ of scaling transformations of the form $T_\lambda \circ (x, t, u) = (\lambda x,\ \lambda^b t,\ u)$. Then the scale-invariant solutions of the equation (55) with the Riemann-Liouville fractional derivative (92) have the form

$$u(x,t) = v(y),\ y = xt^{-\alpha/2} \tag{93}$$

and the substitution (93) reduces the partial integro-differential equation (55) into an ordinary integro-differential equation with the unknown function $v(y)$ given by the following theorem.

THEOREM 3.3.5. *The reduced equation for the scale-invariant solutions in the form* (93) *of the equation* (55) *with the Riemann-Liouville fractional derivative* (92) *is given by*

$$(P_{2/\alpha}^{1-\alpha,\alpha} v)(y) = \mathcal{D}v''(y),\ y > 0 \tag{94}$$

with the left-hand sided Erdélyi-Kober fractional differential operator $P_\delta^{\tau,\alpha}$ *defined for* $0 < \delta,\ n-1 < \alpha \leq n \in \mathbf{N}$ *by*

$$(P_\delta^{\tau,\alpha} g)(y) := \left(\prod_{j=0}^{n-1} (\tau + j - \frac{1}{\delta} y \frac{d}{dy}) \right) (K_\delta^{\tau+\alpha, n-\alpha} g)(y),\ y > 0. \tag{95}$$

Here $(K_{\delta}^{\tau,\alpha}g)(y)$ *is the left-hand sided Erdélyi-Kober fractional integral operator* (84).

The solutions of the equation (94) have been given by Buckwar and Luchko [1] for $\alpha \geq 1$.

THEOREM 3.3.6. *The scale-invariant solutions of the equation* (55) *with the Riemann-Liouville fractional derivative* (92) *in the case* $1 \leq \alpha < 2$ *have the form* $(y = xt^{-\alpha/2})$:

$$u(x,t) = v(y) \;=\; C_1\phi(-\frac{\alpha}{2}, 1, -y/\sqrt{\mathcal{D}}) + C_2\phi(-\frac{\alpha}{2}, 1, y/\sqrt{\mathcal{D}}) \tag{96}$$

with arbitrary constants $C_1,\ C_2$.

Now we consider the case $\alpha > 2$:

THEOREM 3.3.7. *The scale-invariant solutions of the equation* (55) *with the Riemann-Liouville fractional derivative* (92) *in the case* $\alpha > 2,\ \alpha \notin \mathbf{N}$ *have the form* $(y = xt^{-\alpha/2})$:

$$u(x,t) = v(y) = \sum_{j=0}^{[\alpha]} C_j y^{-2+\frac{2}{\alpha}(1+j)}\ {}_2\Psi_1\left[\begin{matrix}(1,1),\ (2-\frac{2}{\alpha}(1+j),2)\\ (\alpha-j,\alpha)\end{matrix}; \mathcal{D}y^{-2}\right], \tag{97}$$

where $C_j,\ 0 \leq j \leq [\alpha]$ *are arbitrary constants and* ${}_p\Psi_q\left[\begin{smallmatrix}(a_1,A_1),\ldots,(a_p,A_p)\\ (b_1,B_1)\ldots(b_q,B_q)\end{smallmatrix}; z\right]$ *is the generalized Wright function* (see [38])*:*

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\ (b_1,B_1)\ldots(b_q,B_q)\end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{i=1}^{q}\Gamma(b_i+B_ik)}\frac{z^k}{k!}. \tag{98}$$

In the case $2 < \alpha = n \in \mathbf{N}$ we have the following result.

THEOREM 3.3.8. *The scale-invariant solutions of the partial differential equation* $(2 < n \in \mathbf{N})$

$$\frac{\partial^n u}{\partial t^n} = \mathcal{D}u_{xx},\ t > 0,\ x > 0,\ \mathcal{D} > 0$$

have the form $(y = x/t^{n/2})$:

$$u(x,t) = \sum_{j=0}^{n-2} C_j y^{-2+\frac{2}{n}(1+j)}\ {}_2\Psi_1\left[\begin{matrix}(1,1),\ (2-\frac{2}{n}(1+j),2)\\ (n-j,n)\end{matrix}; \mathcal{D}y^{-2}\right] + C_{n-1} \tag{99}$$

with arbitrary constants $C_j,\ 0 \leq j \leq n-1$.

Finally, following Luchko and Gorenflo [19], we consider the time-and space-fractional partial differential equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathcal{D}\frac{\partial^\beta u(x,t)}{\partial x^\beta}, \qquad x>0,\ t>0,\ \mathcal{D}>0, \tag{100}$$

where both fractional derivatives are defined in the Riemann-Liouville sense (92).

THEOREM 3.3.9. *The invariants of the group T_λ of scaling transformations under which the equation* (100) *is invariant are given by the expressions*

$$\eta_1(x,t,u) = xt^{-\alpha/\beta},\ \eta_2(x,t,u) = t^{-\gamma}u \tag{101}$$

with an arbitrary constant γ.

THEOREM 3.3.10. *The transformation*

$$u(x,t) \ = \ t^\gamma v(y), \quad y \ = \ xt^{-\alpha/\beta} \tag{102}$$

reduces the partial differential equation of fractional order (100) *to the ordinary differential equation of fractional order of the form*

$$(P^{1+\gamma-\alpha,\alpha}_{\beta/\alpha}v)(y) = \mathcal{D}y^{-\beta}(D_1^{-\beta,\beta}v)(y), \quad y>0. \tag{103}$$

Here the left-hand sided Erdélyi-Kober fractional differential operator $P_\delta^{\tau,\alpha}$ is given by (95) *and the right-hand sided Erdélyi-Kober fractional differential operator $D_\delta^{\tau,\beta}$ is defined for $0<\delta,\ n-1<\beta\le n\in\mathbf{N}$ by*

$$(D_\delta^{\tau,\beta}g)(y) := \left(\prod_{j=1}^{n}(\tau+j+\frac{1}{\delta}y\frac{d}{dy})\right)(I_\delta^{\tau+\beta,n-\beta}g)(y),\ y>0, \tag{104}$$

with the right-hand sided Erdélyi-Kober fractional integral operator

$$(I_\delta^{\tau,\beta}g)(y) := \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^1(1-u)^{\beta-1}u^\tau g(yu^{1/\delta})\,du, & \beta>0,\\ g(y), & \beta=0. \end{cases} \tag{105}$$

Solving the reduced equation we arrive at the following theorem.

THEOREM 3.3.11. *Let*

$$\frac{\beta}{2}\le\alpha<\beta\le 2, \quad n-1<\beta\le n\in\mathbf{N}.$$

Then the scale-invariant (according to the transformation (102) *with $\gamma\ge 0$) solutions of the partial differential equation of fractional order* (100) *have the form*

$$u(x,t) = t^\gamma\sum_{j=1}^{n}C_jv_j(y),\ y=xt^{-\alpha/\beta}, \tag{106}$$

where

$$v_j(y) = y^{\beta-j}\phi((-\alpha, 1+\gamma-\alpha+\frac{\alpha}{\beta}j), (\beta, 1+\beta-j); y^\beta/\mathcal{D}), \qquad (107)$$

the C_j, $1 \le j \le n$ *are arbitrary real constants, and* $\phi((\mu, a), (\nu, b); z)$ *is the generalized Wright function given by* (91).

REMARK 3.3.3. In the case $\beta = 2$ the scale-invariant solutions of the equation (100) can be expressed in terms of the Wright function. Indeed, let us consider the linear combinations of the solutions (107) with $y = xt^{-\alpha/2}$:

$$u_1(x,t) = t^\gamma(\sqrt{\mathcal{D}}v_1(y) + v_2(y)) = t^\gamma\phi(-\frac{\alpha}{2}, 1+\gamma; y/\sqrt{\mathcal{D}}),$$

$$u_2(x,t) = t^\gamma(-\sqrt{\mathcal{D}}v_1(y) + v_2(y)) = t^\gamma\phi(-\frac{\alpha}{2}, 1+\gamma; -y/\sqrt{\mathcal{D}}).$$

These scale-invariant solutions are given in Theorem 3.3.6 in the case $\gamma = 0$.

REMARK 3.3.4. For $0 < \beta \le 1$ the equation (100) has only one solution which is scale-invariant with respect to the transformation (102). This solution has the form $(y = xt^{-\alpha/\beta})$:

$$u(x,t) = t^\gamma v_1(y) = t^\gamma y^{\beta-1}\,\phi((-\alpha, 1+\gamma-\alpha+\frac{\alpha}{\beta}), (\beta, \beta); y^\beta/\mathcal{D}).$$

In the case $\beta = 1$, this function is expressed in terms of the Wright function $(y = xt^{-\alpha})$:

$$u(x,t) = t^\gamma v_1(y) = t^\gamma\phi(-\alpha, 1+\gamma; y/\mathcal{D}).$$

References

[1] E. B u c k w a r, Yu. L u c h k o, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl.* **227** (1998), 81-97.

[2] M. M. D j r b a s h i a n, *Integral Transforms and Representations of Functions in the Complex Plane*. Nauka, Moscow (1966).

[3] M. M. D j r b a s h i a n, *Harmonic Analysis and Boundary Value Problems in the Complex Domain*. Birkhäuser Verlag, Basel (1993).

[4] M. M. D j r b a s h i a n, R. A. B a g i a n, On integral representations and measures associated with Mittag-Leffler type functions. *Izv. Akad. Nauk Armjanskoy SSR, Matematika* **10** (1975), 483-508.

[5] H. E n g l e r, Similarity solutions for a class of hyperbolic integrodifferential equations. *Differential Integral Eqn-s* **10** (1997) 815-840.

[6] M. A. E v g r a f o v, *Asymptotic Estimates and Entire Functions*. Gordon and Breach, New York (1961).

[7] Y. F u j i t a, Integrodifferential equation which interpolates the heat and the wave equations. *Osaka J. Math.* **27** (1990), 309-321, 797-804.

[8] Lj. G a j i ć, B. S t a n k o v i ć, Some properties of Wright's function. *Publ. de l'Institut Mathèmatique, Beograd, Nouvelle Sèr.* **20** (1976), 91-98.

[9] M. G i o n a, H. E. R o m a n, A theory of transport phenomena in disordered systems. *The Chemical Engineering Journal* **49** (1992), 1-10.

[10] R. G o r e n f l o, F. M a i n a r d i, Fractional Calculus: Integral and Differential Equations. In: *Fractals and Fractional Calculus in Continuum Mechanics* (Eds. A. Carpinteri, F. Mainardi). Springer Verlag, Wien - New York (1997), 223-276.

[11] R. G o r e n f l o, F. M a i n a r d i, Fractional calculus and stable probability distributions. *The Archives of Mechanics* **50** (1998), 377-388.

[12] R. G o r e n f l o, F. M a i n a r d i, Signalling problem and Dirichlet-Neumann map for time-fractional diffusion-wave equations. *Matimyás Matematika* **21** (1998), Special Issue, 109-118.

[13] R. G o r e n f l o, Yu. L u c h k o, F. M a i n a r d i, Wright functions as scale-invariant solutions of the diffusion-wave equation. *Journal of Computational and Applied Mathematics* **118** (2000), 175-191.

[14] R. G o r e n f l o, F. M a i n a r d i & H. M. S r i v a s t a v a, Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. In: *Proceedings VIII International Colloquium on Differential Equations, Plovdiv 1997* (Ed. D. Bainov). VSP, Utrecht (1998), 195-202.

[15] R. H i l f e r, Exact solutions for a class of fractal time random walks. *Fractals* **3** (1995), 211-216.

[16] V. K i r y a k o v a, *Generalized Fractional Calculus and Applications*. Longman, Harlow (1994).

[17] E. K r ä t z e l, Integral transformations of Bessel-type. In: *Proc.Conf. Generalized Functions and Operational Calculus*. Publ. House BAS, Sofia (1979), 148-155.

[18] B. Ja. L e v i n, *Distribution of Zeros of Entire Functions*. AMS, Rhode Island, Providence (1980).

[19] Yu. L u c h k o, R. G o r e n f l o, Scale-invariant solutions of a partial differential equation of fractional order. *Fractional Calculus & Applied Analysis* **1**, No 1 (1998), 63-78.

[20] F. M a i n a r d i, On the initial value problem for the fractional diffusion-wave equation. In: *Waves and Stability in Continuous Media* (Eds. S. Rionero and T. Ruggeri). World Scientific, Singapore (1994), 246-251.

[21] F. M a i n a r d i, Fractional diffusive waves in viscoelastic solids. In: *IUTAM Symposium - Nonlinear Waves in Solids* (Eds. J.L. Wagner and F.R. Norwood). ASME/AMR, Fairfield NJ (1995), 93-97.

[22] F. M a i n a r d i, The fundamental solutions for the fractional diffusion-wave equation. *Applied Mathematics Letters* **9**, No 6 (1996), 23-28.

[23] F. M a i n a r d i, Fractional calculus: some basic problems in continuum and statistical mechanics. In: *Fractals and Fractional Calculus in Continuum Mechanics* (Eds. A. Carpinteri and F. Mainardi). Springer Verlag, Wien (1997), 291-348.

[24] F. M a i n a r d i, M. T o m i r o t t i, On a special function arising in the time fractional diffusion-wave equation. In: *Transform Methods & Special Functions, Sofia'94* (Eds. P. Rusev, I. Dimovski and V. Kiryakova). Science Culture Technology Publ., Singapore (1995), 171-183.

[25] O. I. M a r i c h e v, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*. Ellis Horwood, Chichester (1983).

[26] R. M e t z l e r, W. G. G l ö c k l e & T. F. N o n n e n m a c h e r, Fractional model equation for anomalous diffusion. *Physica A* **211** (1994), 13-24.

[27] J. M i k u s i ń s k i, On the function whose Laplace transform is $\exp(-s^{\alpha}\lambda)$, $0 < \alpha < 1$. *Studia Math.* XVIII (1959), 191-198.

[28] R. R. N i g m a t u l l i n, The realization of the generalized transfer equation in a medium with fractal geometry. *Phys. Stat. Sol. B* **133** (1986), 425-430.

[29] P. J. O l v e r, *Applications of Lie Groups to Differential Equations*. Springer Verlag, New York (1986).

[30] R. S. P a t h a k, A general differential equation satisfied by a special function. *Progress of Math.* **6** (1972), 46-50.

[31] A. C. P i p k i n, *Lectures on Viscoelastic Theory*. Springer Verlag, New York (1986).

[32] I. P o d l u b n y, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[33] H. P o l l a r d, The representation of $\exp(-x^{\lambda})$ as a Laplace integral. *Bull. Amer. Math. Soc.* **52** (1946), 908-910.

[34] J. P r ü s s, *Evolutionary Integral Equations and Applications*. Birkhäuser, Basel (1993).

[35] A. S a i c h e v, G. Z a s l a v s k y, Fractional kinetic equations: solutions and applications. *Chaos* **7** (1997), 753-764.

[36] S. G. S a m k o, A. A. K i l b a s & O. I. M a r i c h e v, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, New York (1993).

[37] W. R. S c h n e i d e r, W. W y s s, Fractional diffusion and wave equations. *J. Math. Phys.* **30** (1989), 134-144.

[38] H. M. S r i v a s t a v a, K. C. G u p t a, S. P. G o y a l, *The H-Functions of One and Two Variables with Applications*. South Asian Publishers, New Delhi - Madras (1982).

[39] B. S t a n k o v i ć, On the function of E.M. Wright. *Publ. de l'Institut Mathèmatique, Beograd, Nouvelle Sèr.* **10** (1970), 113-124.

[40] G. N. W a t s o n, *A Treatise on the Theory of Bessel Functions*, 2nd Ed. Cambridge Univ. Press, London (1944).

[41] E. M. W r i g h t, On the coefficients of power series having exponential singularities. *Journal London Math. Soc.* **8** (1933), 71-79.

[42] E. M. W r i g h t, The asymptotic expansion of the generalized Bessel function. *Proc. London Math. Soc. (Ser. II)* **38** (1935), 257-270.

[43] E. M. W r i g h t, The asymptotic expansion of the generalized hypergeometric function. *Journal London Math. Soc.* **10** (1935), 287-293.

[44] E. M. W r i g h t, The generalized Bessel function of order greater than one. *Quart. J. Math., Oxford ser.* **11** (1940), 36-48.

[45] W. W y s s, Fractional diffusion equation. *J. Math. Phys.* **27** (1986), 2782-2785.

[1] *Dept. of Mathematics and Computer Science*
Free University of Berlin
Arnimallee 3
D-14195 Berlin – GERMANY
e-mail: gorenflo@math.fu-berlin.de

[2] *Dept. of Business Informatics*
Europe University Frankfurt (Oder)
POB 1786
D-15207 Frankfurt (Oder) – GERMANY
e-mail: loutchko@euv-frankfurt-o.de

[3] *Dipartimento di Fisica*
Università di Bologna and INFN
Sezione di Bologna, Via Irnerio 46
I-40126 Bologna – ITALY
e-mail: mainardi@bo.infn.it

Received: August 13, 1999