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Random Walk Models for Space-Fractional Diffusion Processes

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Contents

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Abstract

By space-fractional (or Lévy-Feller) diffusion processes we mean the processes governed by a generalized diffusion equation which generates all Lévy stable probability distributions with index α ($0 < \alpha \leq 2$), including the two symmetric most popular laws, Cauchy ($\alpha = 1$) and Gauss ($\alpha = 2$). This generalized equation is obtained from the standard linear diffusion equation by replacing the second-order space derivative with a suitable fractional derivative operator, defined as inverse of the Feller potential (a generalization of the Riesz potential). In this paper, excluding the singular case $\alpha = 1$ and based on the Grünwald-Letnikov approach to the fractional derivative, we propose to approximate these processes by random walk models, discrete in space and time. It is proved that for properly scaled transition to the limit of vanishing space and time steps there is convergence to all (symmetric and nonsymmetric) stable distributions (evolving in time). All stable distributions are thus obtained, with the exception of the Cauchy distribution which resists to this treatment.

Keywords: Stochastic processes, diffusion equations, fractional calculus, Riesz-Feller potentials, stable probability distributions, random walks.

1991 Mathematics Subject Classification: 26A33, 33E20, 44A20, 45K05, 60E07, 60J15, 60J60, 60J65

1. Introduction

William Feller in his pioneering paper of 1952 [7] investigated the semigroups generated by the one-dimensional pseudo-differential operators arising by inversion of linear combinations of left and right hand sided Riemann-Liouville-Weyl operators (in case of both coefficients being equal this amounts to inversion of Riesz potential operators). These semigroups can be interpreted as descriptions of space-fractional diffusion processes, evolving in time, which generalize the Gauss process (of classical diffusion or heat conduction). Viewed from the probabilistic standpoint these semigroups represent stable distributions (scaled by a similarity variable combining space and time) in the sense of Paul Lévy [21-23]. Honoring both Lévy and Feller, we call the described stochastic processes (which are of Markov type) "Lévy-Feller processes". These processes are characterized by two parameters α (positive but no greater than 2) and θ (suitably restricted) for the order and the "skewness" of the spatial pseudo-differential operator. We give (in our notation) the essentials of Feller's theory in Section 3 of this paper.

Random walk models, discrete in space and time, for the standard diffusion equation and its generalizations to presence of drift and spatially varying diffusivity have a long history. Such models are not only valuable from the conceptual point of view for visualizing what diffusion means but also for numerical calculations, either as Monte Carlo simulation of particle paths in a diffusion process or as discrete imitation of the process in form of redistribution (from one time level to the next) of clumps of an extensive quantity (across the spatial grid points). Such extensive quantities, *e.g.* , are mass or charge, but in our context also sojourn probability can (at least formally) be considered as an extensive quantity. For orientation on such aspects and for examples let us quote Feller [8], Prabhu [29], Gorenflo [13-14], Gorenflo & Niedack [16].

In Section 2 we clarify our concepts and introduce notations by laying out the standard random walk model for the classical diffusion equation. This model and later our models for the Lévy-Feller processes are constructed as two-level (in time) three point (in space) difference schemes.

In Section 4 we construct random walk models, based on the Grünwald-Letnikov discretization of the fractional derivatives occurring in the spatial pseudo-differential operator, for the Lévy-Feller processes (omitting the Cauchy-process that corresponds to the value 1 of the order parameter α). In the case $\alpha = 2$ the three-point standard random walk model for the classical diffusion equation is recovered, but in all other cases arbitrarily large jumps (with power-like decay of far-away transition probabilities) must be admitted. We exhibit in detail how these models work, paying special attention to what happens if extremal values of the skewness parameter are taken.

In Section 5 we show that indeed our random walk model does "converge" to the corresponding Lévy-Feller process. More precisely, we show that for a particle starting at the origin the characteristic function of the random walk's sojourn probabilities at a fixed positive time tends to the Fourier transform of the Green function of the corresponding Lévy-Feller process. By a theorem of general probability theory then the convergence of the cumulative distribution functions to that of the continuous process follows.

The argumentation in this Section 5 can be inverted. In fact, let us be given by God our random walk model, and assume we know nothing of Lévy probability distributions. Then, just by properly scaled passage to the limit of an infinitely fine grid we obtain these. And gratis (because the discrete probabilities are all non-negative they cannot become negative in the limit) we get that the limiting densities are everywhere non-negative, for all values of the order parameter α between 0 and 2 (the value 1 omitted). Just by manipulation of these random walk models we then obtain these probability distributions and the corresponding processes, without need of the theory of Riesz and Feller potentials and their inversion and without need of the method of positive definite functions. We thus have an alternative way of solving a problem that surmounted Cauchy's capabilities who in 1853 had considered the functions $\exp(-|\kappa|^{\alpha})$ as candidates of Fourier transforms of probability densities but could only prove them to have this property in the special cases $\alpha = 1\,,\,2$. Bochner [3] has given an elegant proof that for the full range of α positive but no greater than 2 the inverse Fourier transforms of these functions is everywhere non-negative, hence a probability density. For this he used the theory of positive definite functions that we can avoid. A well readable account of Bochner's method is reproduced by Montroll & West [27].

For the sake of convenience and completeness, we provide our notation for the Fourier transform in Appendix A and the essential notions of Lévy stable probability distributions in Appendix B.

Let us finally remark that, wanting our paper to be accessible to various kinds of people working in applications (*e.g.* physicists, chemists, theoretical biologists, economists, engineers) we have deliberately and consciously as far as possible avoided the languages of functional analysis and of abstract measure-theoretic probability. In some places we have used vague phrases like "for a sufficiently well behaved function" instead of constructing a stage of precisely defined spaces of admissible functions. We kindly ask specialists of these fields of pure mathematics to forgive us. Our paper is written in a way that makes it easy to fill in details of precision which in their opinion might be lacking.

2. The Standard Diffusion Equation

For the standard diffusion equation it is well known that the fundamental solution of the *Cauchy* problem provides the space *probability density function* (*pdf*) for the Gaussian or normal distribution, whose variance is proportional to time. For convenience, let us derive this result, making use of the Fourier transform, using the notation introduced in Appendix A.

The *Cauchy* problem for the standard diffusion equation reads

$$\begin{cases} \dfrac{\partial}{\partial t}\, u(x,t) = \mathcal{D}\, \dfrac{\partial^2}{\partial x^2}\, u(x,t)\,, \quad x \in \mathbb{R}\,, \quad t \in \mathbb{R}_0\,, \\ u(x,0^+) = g(x)\,, \end{cases} \tag{2.1}$$

where $\mathcal{D} > 0$ denotes the diffusion coefficient (of dimension $L^2 T^{-1}$), $\mathbb{R}$ ($\mathbb{R}_0$) is the set of the real (non negative) numbers, and $g(x)$ is a sufficiently well-behaved real function defined on $\mathbb{R}$. From now on, for the sake of convenience, we agree to put $\mathcal{D} = 1$.

Then, the application of the Fourier transform to (2.1) leads us to solve a simple differential equation of the first order in time, so the transformed solution reads

$$\hat{u}(\kappa,t) = \hat{g}(\kappa)\, \mathrm{e}^{-t\,|\kappa|^2}\,, \quad k \in \mathbb{R}\,. \tag{2.2}$$

Then, introducing

$$\mathcal{G}_c^d(x,t) \div \widehat{\mathcal{G}_c^d}(\kappa,t) = \mathrm{e}^{-t\,|\kappa|^2}\,, \tag{2.3}$$

the required solution is provided by inversion in terms of the space convolution

$$u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c^d(\xi,t)\, g(x-\xi)\, d\xi\,, \tag{2.4}$$

where

$$\mathcal{G}_c^d(x,t) = \frac{1}{2\sqrt{\pi}}\, t^{-1/2}\, \mathrm{e}^{-x^2/(4\,t)}\,. \tag{2.5}$$

Here $\mathcal{G}_c^d(x,t)$ represents the fundamental solution (or Green function) of the Cauchy problem, since *formally* it corresponds to $g(x) = \delta(x)$; the upper index d refers to (standard) *diffusion* whereas the lower index c refers to *Cauchy* problem.

The interpretation of such Green function in probability theory is straightforward since we easily recognize

$$\mathcal{G}_c^d(x,t) = p_G(x;\sigma(t)) \,, \quad \sigma^2 = 2t \,, \tag{2.6}$$

where $p_G(x;\sigma)$ denotes the well-known *Gauss* or *normal pdf* whose moment of the second order, the *variance*, is σ^2. For convenience the Gauss distribution has been recalled in Appendix B, see (B.3) and (B.6), in the framework of stable distributions.

The common numerical approach for the standard diffusion equation is known to be based on the finite-difference method where the first time derivative and the second space derivative are obtained by a finite-difference scheme.

For this purpose we discretize space and time by grid points

$$x_j = j\,h \,, \quad h > 0 \,, \quad j = 0, \pm 1, \pm 2, \dots \tag{2.7}$$

and time instants

$$t_n = n\,\tau \,, \quad \tau > 0 \,, \quad n = 0, 1, 2, \dots \tag{2.8}$$

The dependent variable is then discretized by introducing $y_j(t_n)$ as

$$y_j(t_n) = \int_{x_j - h/2}^{x_j + h/2} u(x, t_n)\, dx \approx h\, u(x_j, t_n) \,. \tag{2.9}$$

With the quantities $y_j(t_n)$ so intended, we replace the standard diffusion equation (2.1) by the finite-difference equation

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = \frac{y_{j+1}(t_n) - 2y_j(t_n) + y_{j-1}(t_n)}{h^2} \,, \tag{2.10}$$

accepting that for positive n in (2.9) we have approximate instead of exact equality. If we are interested to obtain the fundamental solution (the Green function), we must equip (2.10) with the initial condition

$$y_j(0) = \delta_{j\,0} \,, \tag{2.11}$$

since the Kronecker symbol represents the discrete counterpart of the Dirac delta function. Otherwise, the initial condition is

$$y_j(0) = \int_{x_j - h/2}^{x_j + h/2} g(x)\, dx \, . \tag{2.12}$$

This approach can be interpreted as a discrete (in space and time) *redistribution process* of some extensive quantity, *e.g.* mass or a sojourn probability.

In the first case the $y_j(t_n)$ are imagined as clumps of mass, sitting at grid points $x = x_j$ in instants $t = t_n$, which collect approximatively the total mass in the interval $x_j - h/2 < x \le x_j + h/2$.

In the second case, by a suitable normalization, the $y_j(t_n)$ may be interpreted as the probability of sojourn in point x_j at time t_n for a particle making a *random walk* on the spatial grid in discrete instants. From now on, we agree to pursue the probabilistic point of view. We refer to (2.10-11) as the standard random walk model for the Gaussian process.

When time proceeds from $t = t_n$ to $t = t_{n+1}$, the sojourn-probabilities are redistributed according to the general rule

$$y_j(t_{n+1}) = \sum_{k=-\infty}^{\infty} p_k \, y_{j-k}(t_n) \, , \quad j \in \mathbf{Z} \, , \quad n \in \mathbb{N}_0 \, , \tag{2.13}$$

where the p_k denote suitable *transfer coefficients*, which represent the probability of transition from x_{j-k} to x_j (likewise from x_j to x_{j+k}).

The transfer coefficients are to be found consistently with the finite-difference equation (2.10) equipped with the proper initial condition, (2.11) or (2.12).

The process is conceived as both spatially homogeneous (the probability p_k of jumping from a point x_j to a point x_{j+k} not depending on j) and time stationary (the p_k not depending on n), as is advised when considering our Cauchy problem and the definition of the difference operators.

We note that some relevant conditions must be satisfied in (2.13) to ensure that we have a *conservative* and *non-negativity preserving* redistribution process. In fact, the discrete variable y_j is subject to the conditions

$$\sum_{j=-\infty}^{+\infty} y_j(t_n) = \sum_{j=-\infty}^{+\infty} y_j(0) \, , \quad y_j(t_n) \ge 0 \, , \quad \forall t_n \, . \tag{2.14}$$

whereas the transfer coefficients satisfy the conditions

$$\sum_{k=-\infty}^{\infty} p_k = 1\,, \quad p_k \geq 0\,, \quad k = 0\,, \pm 1\,, \pm 2\,, \ldots \tag{2.15}$$

The transfer coefficients are easily deduced from (2.10) and (2.13): they turn out to be

$$\begin{cases} p_0 = 1 - 2\dfrac{\tau}{h^2}\,, \\ p_{\pm 1} = \dfrac{\tau}{h^2}\,, \\ p_{\pm k} = 0\,, \quad k = 2, 3 \ldots\,, \end{cases} \tag{2.16}$$

subject to the condition

$$0 < \frac{\tau}{h^2} \leq \frac{1}{2}\,. \tag{2.17}$$

The scheme (2.16) means that for approximation of standard diffusion only jumps of one step to the right or one to the left or jumps of width zero occur.

In Section 4 we shall assign suitably modified transition probabilities p_k, so that equation (2.13) will become a random walk model for the Lévy-Feller processes (the case $\alpha = 1$ left aside).

3. The Feller Space-Fractional Diffusion Equation

Feller [7] considered the problem of generating all the stable probability distributions through the Green function of the Cauchy problem for a suitable fractional diffusion equation.

Here we provide a completely revised version of Feller's genial paper and we complement the analysis by a finite-difference approach, which is original as far as we know. Our approach allows us to generalize the classical random walk model of the standard diffusion equation to the Feller fractional diffusion equation and to compute numerically all the stable, non-Gaussian densities, with the exception of those of Cauchy-type, which resist to this treatment.

Feller's essential idea is to replace the second-order space derivative of the standard diffusion equation with a special pseudo-differential operator whose symbol is required to be the logarithm of the characteristic function of the generic *Lévy*, strictly stable distribution of index α $(0 < \alpha \leq 2)$ and skewness θ (suitably restricted), according to the notation introduced in Appendix B, see (B.8-9).

Here, Feller's operator is denoted by D_θ^α and its symbol reads

$$\widehat{D_\theta^\alpha}(\kappa) := \log\left[\hat{p}_\alpha(\kappa;\theta)\right] = -|\kappa|^\alpha \, \mathrm{e}^{\, i\,(\mathrm{sign}\,\kappa)\,\theta\pi/2} \,, \quad 0 < \alpha \le 2 \,, \tag{3.1}$$

where θ assumes restricted values, depending on α, as follows,

$$\begin{cases} |\theta| \le \alpha \,, & \text{if } \; 0 < \alpha < 1 \,, \\ |\theta| \le 2 - \alpha \,, & \text{if } \; 1 < \alpha < 2 \,, \\ \theta = 0 \,, & \text{if } \quad \alpha = 1\,,\, 2 \,. \end{cases} \tag{3.2}$$

By the *Feller space-fractional diffusion* equation we thus mean the linear evolution equation

$$\frac{\partial\, u}{\partial t} = D_\theta^\alpha \, u \,, \quad u = u(x,t;\alpha,\theta) \,, \quad x \in \mathbb{R} \,, \quad t \in \mathbb{R}_0^+ \,, \tag{3.3}$$

where $0 < \alpha \le 2$ and θ restricted as in (3.2).

Also here, for the sake of convenience, we agree to put equal to one the coefficient (of dimension $L^\alpha T^{-1}$) preceeding the space derivative.

As for the standard diffusion equation, the *Cauchy* problem may be solved by making use of the Fourier transform and introducing the corresponding *Green* function, since the Fourier representation of the space-fractional derivative is known. Here we write

$$u(x,t;\alpha,\theta) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t;\alpha,\theta)\, g(x-\xi)\, d\xi \,, \tag{3.4}$$

where $g(x)$ is a sufficiently well-behaved real function defined in $\mathbb{R}$, which provides the initial data $u(x,0^+)$, and the *Green* function $\mathcal{G}_c(x,t;\alpha;\theta)$ is the fundamental solution of (3.3), formally obtained when $g(x) = \delta(x)$.

As a consequence of the definition (3.1), the Fourier transform of the Green function turns out to be

$$\widehat{\mathcal{G}_c}(\kappa,t;\alpha,\theta) = \exp\left[t\, \widehat{D_\theta^\alpha}(\kappa)\right] \,. \tag{3.5}$$

By inversion of (3.5) we obtain all the stable distributions evolving in time. In other words, the parameter t plays the role of a scale factor, and the Green function turns out to be

$$\mathcal{G}_c(x,t;\alpha,\theta) = t^{-1/\alpha}\, p_\alpha(x\, t^{-1/\alpha}, \theta) \,. \tag{3.6}$$

Therefore (3.6) provides the required interpretation in probability theory for the Feller diffusion equation, which generalizes the corresponding (2.5) or (2.6), in terms of the Gaussian law, for the standard diffusion equation.

In general, when $\theta = 0$, the related stable distributions are symmetric and the corresponding space derivative is said to be *symmetric.*

In the symmetric case we can adopt the notation introduced by Zaslavski, see *e.g.* [30],

$$D_0^\alpha \, \phi(x) \, := \, \frac{d^\alpha \, \phi}{d|x|^\alpha} \, \div \, -|\kappa|^\alpha \,, \qquad 0 < \alpha \le 2 \,. \tag{3.7}$$

We note for $\alpha = 2$

$$-|\kappa|^2 = -\kappa^2 = (-i\kappa)^2 \, \div \, D_0^2 := \frac{d^2}{d|x|^2} = \frac{d^2}{dx^2} \,,$$

whereas, in general, following Feller [7],

$$-|\kappa|^\alpha = - \left(|\kappa|^2\right)^{\alpha/2} \, \div \, D_0^\alpha := \frac{d^\alpha}{d|x|^\alpha} = - \left(-\frac{d^2}{dx^2}\right)^{\alpha/2} \,. \tag{3.8}$$

We thus recognize that the operator D_0^α is related to a power of the positive definitive operator $-\frac{d^2}{dx^2}$ and must not be confused with a power of the first order differential operator $\frac{d}{dx}$ for which the symbol is $-i\kappa$.

Let us now express more properly the operator $-D_\theta^\alpha$ as inverse of a suitable integral operator I_θ^α, whose symbol is required to be $|\kappa|^{-\alpha} \, \mathrm{e}^{-i \, (\mathrm{sign}\, \kappa) \, \theta\pi/2}$, so we may write

$$D_\theta^\alpha := -I_\theta^{-\alpha} \,. \tag{3.9}$$

This integral operator was found by Feller generalizing the approach by Marcel Riesz to Fractional Calculus, and it is referred to as *Feller potential* by Samko, Kilbas & Marichev [31]*

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* We must note that in his original paper Feller [7] used a skewness parameter δ different from our θ; the symbol of the Feller potential is

$$\left(|\kappa| \, \mathrm{e}^{-i \, (\mathrm{sign}\, \kappa) \, \delta}\right)^{-\alpha} \,, \quad \text{so} \quad \delta = -\frac{\pi}{2} \, \frac{\theta}{\alpha} \,, \quad \theta = -\frac{2}{\pi} \, \alpha\delta \,.$$

Feller and Samko, Kilbas & Marichev thus use I_δ^α where their δ is related to our θ as above.

Using our notation, the *Feller potential* reads

$$I_\theta^\alpha\,\phi(x) = c_-(\alpha,\theta)\,I_+^\alpha\,\phi(x) + c_+(\alpha,\theta)\,I_-^\alpha\,\phi(x)\,, \tag{3.10}$$

where, if $0 < \alpha < 2\,,\ \alpha \neq 1\,,$

$$\begin{cases} c_+(\alpha,\theta) = \dfrac{\sin\left[\frac{\pi}{2}(\alpha-\theta)\right]}{\sin(\alpha\pi)}\,, \\ c_-(\alpha,\theta) = \dfrac{\sin\left[\frac{\pi}{2}(\alpha+\theta)\right]}{\sin(\alpha\pi)}\,, \end{cases} \tag{3.11}$$

and, by passing to the limit (with $\theta = 0$) $c_+(2,0) = c_-(2,0) = -1/2\,.$ In (3.10) the operators $I_\pm^\alpha$ denote the *Riemann-Liouville integrals* (by some people called *Weyl integrals*),

$$\begin{cases} I_+^\alpha\,\phi(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \\ I_-^\alpha\,\phi(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{x}^{+\infty}(\xi-x)^{\alpha-1}\,\phi(\xi)\,d\xi\,. \end{cases} \tag{3.12}$$

We note that whereas the coefficients $c_\pm$ can loose their meaning when α is an integer, the Riemann-Liouville integral operators $I_\pm^\alpha$ are well defined in their action on rapidly decreasing integrable functions, for any $\alpha \geq 0\,,$ being set equal to the identity operator when $\alpha = 0\,,$ for convenience (justified by passage to the limit $\alpha \to 0$).

In the particular case $\theta = 0$ we get

$$c_+(\alpha,0) = c_-(\alpha,0) = \frac{1}{2\cos(\pi\alpha/2)}\,, \tag{3.13}$$

and thus we recover the *Riesz potential*, see *e.g.* [31]

$$I_0^\alpha\,\phi(x) := \frac{1}{2\,\Gamma(\alpha)\cos(\pi\alpha/2)}\int_{-\infty}^{+\infty}|x-\xi|^{\alpha-1}\,\phi(\xi)\,d\xi\,. \tag{3.14}$$

The Riesz and the Feller potentials are well defined if the index is located in the range $(0\,,\,1)\,,$ and we have the semigroup property,

$$I_\theta^{\alpha+\beta} = I_\theta^\alpha I_\theta^\beta\,,\quad 0<\alpha<1\,,\quad 0<\beta<1\,,\quad \alpha+\beta<1\,.$$

Then, following Feller, we define by analytic continuation the pseudo-differential operator (3.9) in the whole range $0 < \alpha \le 2$ as

$$D_\theta^\alpha := -\left[c_+(\alpha,\theta)\, I_+^{-\alpha} + c_-(\alpha,\theta)\, I_-^{-\alpha}\right]\,, \quad 0 < \alpha \le 2\,, \tag{3.15}$$

where $I_+^{-\alpha}$ and $I_-^{-\alpha}$ are the inverses of the operators I_+^α and I_-^α, respectively. For integral representations of the above operators, see [31]. We have

$$I_\pm^{-\alpha} = \begin{cases} \pm \dfrac{d}{dx}\, I_\pm^{1-\alpha}\,, & \text{if} \quad 0 < \alpha \le 1\,, \\[2ex] \dfrac{d^2}{dx^2}\, I_\pm^{2-\alpha}\,, & \text{if} \quad 1 < \alpha \le 2\,. \end{cases} \tag{3.16}$$

We note that we define the space-fractional derivative with sign opposite to the inverse of the Feller potential, to ensure that for $\alpha = 2$ we recover the second space derivative of the standard diffusion equation and, consequently, the *Gauss pdf* .

In fact, for $\alpha = 2$ and $\theta = 0$, we get

$$c_+(2,0) = c_-(2,0) = -1/2\,, \tag{3.17}$$

so from (3.15-17) we easily recognize that

$$D_0^2 = -I_0^{-2} = \frac{1}{2}\left(I_+^{-2} + I_-^{-2}\right) = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{d^2}{dx^2}\right) = \frac{d^2}{dx^2}\,. \tag{3.18}$$

In this way we are also in agreement with [30] for $0 < \alpha < 2$ and $\theta = 0$.

Here we shall not consider the singular case $\alpha = 1$, which however is treated by Feller [7] through the Hilbert transform and the Poisson integral.

For later use we find it convenient to return to the "weight" coefficients $c_\pm(\alpha,\theta)$ in order to outline some properties along with some particular expressions, which can be easily obtained from (3.11) with the restrictions on θ given in (3.2). We obtain

$$c_\pm \begin{cases} \ge 0\,, & \text{if } 0 < \alpha < 1\,, \\ \le 0\,, & \text{if } 1 < \alpha \le 2\,, \end{cases} \tag{3.19}$$

and

$$c_+ + c_- = \frac{\cos(\theta\pi/2)}{\cos(\alpha\pi/2)} \begin{cases} > 0\,, & \text{if } 0 < \alpha < 1\,, \\ < 0\,, & \text{if } 1 < \alpha \le 2\,. \end{cases} \tag{3.20}$$

In the *extremal* cases we find

$$0 < \alpha < 1\,, \begin{cases} c_+ = 1\,,\ c_- = 0\,, & \text{if}\ \ \theta = -\alpha\,, \\ c_+ = 0\,,\ c_- = 1\,, & \text{if}\ \ \theta = +\alpha\,, \end{cases} \tag{3.21a}$$

and

$$1 < \alpha < 2\,, \begin{cases} c_+ = 0\,,\ c_- = -1\,, & \text{if}\ \ \theta = -(2-\alpha)\,, \\ c_+ = -1\,,\ c_- = 0\,, & \text{if}\ \ \theta = +(2-\alpha)\,. \end{cases} \tag{3.21b}$$

4. Random Walks for Lévy-Feller Diffusion Processes

We now start the original part of our work in providing a finite-difference approach to approximate and in the limit to find the Green function of the Cauchy problem for the Feller space-fractional diffusion equation (3.3), excluding the singular case $\alpha = 1$. Such an approach is better interpreted as a peculiar random-walk model (or redistribution process), discrete in space and time, so extending the classical argument for the standard diffusion equation.

The *essential idea* is to approximate the inverse operators $I_\pm^{-\alpha}$ entering the expression (3.15) of the Feller derivative by the Grünwald-Letnikov scheme, on which the reader can inform himself in the treatises on fractional calculus, see *e.g.* Oldham & Spanier [28], Samko, Kilbas & Marichev [31], Miller & Ross [26], or in the recent review article by Gorenflo [15].

If h denotes a "small" positive step-length, the inverse of the Riemann-Liouville integrals can be obtained in the limit

$$I_\pm^{-\alpha} = \lim_{h\to 0} {}_hI_\pm^{-\alpha}\,, \tag{4.1}$$

where ${}_hI_\pm^{-\alpha}$ denote the approximating Grünwald-Letnikov operators, which read

(a) $\quad 0 < \alpha \le 1\,,$

$$ {}_hI_\pm^{-\alpha}\,\phi(x) := \frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k}\,\phi(x \mp kh)\,, \tag{4.2a}$$

(b) $\quad 1 < \alpha \le 2\,,$

$$ {}_hI_\pm^{-\alpha}\,\phi(x) := \frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k}\,\phi(x \mp (k-1)h)\,. \tag{4.2b}$$

In particular the derivatives of order 1 and 2 are obtained from the limits

$$\frac{d}{dx}\,\phi(x) = \begin{cases} +\,I_{+}^{-1}\,\phi(x) = \displaystyle\lim_{h\to 0} \frac{\phi(x)-\phi(x-h)}{h}\,, \\ -\,I_{-}^{-1}\,\phi(x) = \displaystyle\lim_{h\to 0} \frac{\phi(x+h)-\phi(x)}{h}\,, \end{cases}$$

and

$$\frac{d^2}{dx^2}\,\phi(x) = \begin{cases} I_{+}^{-2}\,\phi(x) = \displaystyle\lim_{h\to 0} \frac{\phi(x+h)-2\phi(x)+\phi(x-h)}{h^2}\,, \\ I_{-}^{-2}\,\phi(x) = \displaystyle\lim_{h\to 0} \frac{\phi(x-h)-2\phi(x)+\phi(x+h)}{h^2}\,. \end{cases}$$

We note the shift in the index k at the r.h.s. of (4.2b) if $1 < \alpha \le 2$; this shift is a consistent approximation for sufficiently smooth functions, as shown by the standard second order difference quotient for the second derivative. We can understand the operator in (4.2b) as similar to that in (4.2a) but acting on $\phi(x \pm h)$ rather than on $\phi(x)$.

Usually, the standard Grünwald-Letnikov scheme is given by (4.2a), for any $\alpha > 0$ without any shift; with this (”universal”) choice we would find the following expressions for the second derivative,

$$\frac{d^2}{dx^2}\,\phi(x) = \begin{cases} I_{+}^{-2}\,\phi(x) = \displaystyle\lim_{h\to 0} \frac{\phi(x)-2\phi(x-h)+\phi(x-2h)}{h^2}\,, \\ I_{-}^{-2}\,\phi(x) = \displaystyle\lim_{h\to 0} \frac{\phi(x)-2\phi(x+h)+\phi(x+2h)}{h^2}\,, \end{cases}$$

which are non-consistent with the symmetric character of the space derivative in the standard diffusion equation.

The reader can now understand the importance of our shift in order to generalize the standard diffusion equation by introducing a space-fractional derivative along with a ”clever” use of the Grünwald-Letnikov approximation. A similar shift in the Grünwald-Letnikov approximation. has been profitably used by Vu Kim Tuan and Gorenflo for improving the order of accuracy in their paper [35].

Discretizing all the variables as for the standard diffusion equation, see (2.7-9), we replace the Feller diffusion equation (3.3) by the finite-difference equation

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = {}_hD_\theta^\alpha\, y_j(t_n)\,, \tag{4.3}$$

where the *difference operator* ${}_hD_\theta^\alpha$ reads

$$ {}_hD_\theta^\alpha \, y_j(t_n) = - \left[c_+ \, {}_hI_+^{-\alpha} \, y_j(t_n) + c_- \, {}_hI_-^{-\alpha} \, y_j(t_n) \right] , \tag{4.4} $$

with $0 < \alpha \le 2$, $\alpha \ne 1$. Here the coefficients $c_\pm = c_\pm(\alpha, \theta)$ are given by (3.11) with θ restricted as in (3.2), and the operators ${}_hI_\pm^{-\alpha}$ turn out to be, in view of (4.2a) and (4.2b)

$$ {}_hI_\pm^{-\alpha} y_j = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j \mp k} \, , \qquad 0 < \alpha < 1 \, , \tag{4.5a} $$

$$ {}_hI_\pm^{-\alpha} y_j = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y_{j \pm 1 \mp k} \, , \qquad 1 < \alpha \le 2 \, . \tag{4.5b} $$

We again note the shifted indices j on the r.h.s. of (4.5b), if $1 < \alpha \le 2$.

The finite-difference equation (4.3) is subject to the initial condition (2.11) or (2.12) depending whether we are interested to find the Green function of the Cauchy problem or to find the solution with initial data $g(x)$.

Let us now construct our random walk model for the Lévy-Feller process, *i.e.* let us determine the transition probabilities in the general expression (2.13), subject to the conditions (2.14-15), by our "clever" use of the Grünwald-Letnikov approximating operators in the finite-difference equation (4.3). For this purpose we must keep distinct the two cases

$$ (a) \quad 0 < \alpha < 1 \, , \quad |\theta| \le \alpha \, , \qquad (b) \quad 1 < \alpha \le 2 \, , \quad |\theta| \le 2 - \alpha \, . $$

<u>In the case (a)</u>, using (4.4) and (4.5a) the finite-difference equation (4.3) turns out to be

$$ \begin{aligned} y_j(t_{n+1}) &= y_j(t_n) \\ &- \frac{\tau}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \left[c_+ \, y_{j-k}(t_n) + c_- \, y_{j+k}(t_n) \right] . \end{aligned} \tag{4.6a} $$

From comparison between (4.6a) and (2.13), and accounting for (3.20), the transition probabilities turn out to be

$$ \begin{cases} p_0 = 1 - \dfrac{\tau}{h^\alpha} (c_+ + c_-) = 1 - \dfrac{\tau}{h^\alpha} \dfrac{\cos(\theta\pi/2)}{\cos(\alpha\pi/2)} \, , \\ p_{\pm k} = (-1)^{k+1} \dfrac{\tau}{h^\alpha} \dbinom{\alpha}{k} c_\pm = \dfrac{\tau}{h^\alpha} \left| \dbinom{\alpha}{k} \right| c_\pm \, , \quad k = 1, 2, \ldots \end{cases} \tag{4.7a} $$

where the $c_\pm$ are given by (3.11).

We see that the summation condition in (2.15) is fulfilled. In fact,

$$\sum_{k=-\infty}^{\infty} p_k = p_0 + \sum_{k=1}^{\infty}(p_k + p_{-k}) = 1 - \frac{\tau}{h^\alpha}(c_+ + c_-)\sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} = 1\,.$$

We have to check the non-negativity condition as well. Observe that all $p_{\pm k} \geq 0\,,\ k \in \mathbb{N}\,,$ since $c_\pm \geq 0$ in the case (a), whereas $p_0 \geq 0$ if and only if the space step h and the time step τ are related according to

$$0 < \frac{\tau}{h^\alpha} \leq \frac{\cos(\alpha\pi/2)}{\cos(\theta\pi/2)}\,, \qquad 0 < \alpha < 1\,, \quad |\theta| \leq \alpha\,. \tag{4.8a}$$

In the case (b), using (4.4) and (4.5b) the finite-difference equation (4.3) turns out to be

$$\begin{aligned} y_j(t_{n+1}) &= y_j(t_n) \\ &\quad - \frac{\tau}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\left[c_+\, y_{j+1-k}(t_n) + c_-\, y_{j-1+k}(t_n)\right]. \end{aligned} \tag{4.6b}$$

From comparison between (4.6b) and (2.13), and accounting for (3.11), (3.20), the transition probabilities turn out to be

$$\begin{cases} p_0 = 1 + \dfrac{\tau}{h^\alpha}\dbinom{\alpha}{1}(c_+ + c_-) = 1 - \dfrac{\tau}{h^\alpha}\,\alpha\,\dfrac{\cos(\theta\pi/2)}{|\cos(\alpha\pi/2)|}\,, \\ p_{\pm 1} = -\dfrac{\tau}{h^\alpha}\left[\dbinom{\alpha}{2}c_\pm + c_\mp\right] = \dfrac{\tau}{h^\alpha}\left[\left|\dbinom{\alpha}{2}\right|\,|c_\pm| + |c_\mp|\right] \\ p_{\pm k} = (-1)^k\,\dfrac{\tau}{h^\alpha}\dbinom{\alpha}{k+1}c_\pm = \dfrac{\tau}{h^\alpha}\left|\dbinom{\alpha}{k+1}\right|\,|c_\pm|\,,\ k = 2, 3, \dots \end{cases} \tag{4.7b}$$

where the $c_\pm$ are given by (3.11).

We see that the summation condition in (2.15) is fulfilled. In fact,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} p_k &= p_0 + p_1 + p_{-1} + \sum_{k=2}^{\infty}(p_k + p_{-k}) \\ &= 1 - \frac{\tau}{h^\alpha}(c_+ + c_-)\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} = 1\,. \end{aligned}$$

We have to check the non-negativity condition as well. Observe that all $p_{\pm k} \geq 0$, $k \in \mathbb{N}$, since $c_\pm \leq 0$ in the case (b), whereas $p_0 \geq 0$ if and only if the space step h and the time step τ are related according to

$$0 < \frac{\tau}{h^\alpha} \leq \frac{1}{\alpha} \frac{|\cos(\alpha\pi/2)|}{\cos(\theta\pi/2)}, \quad 1 < \alpha \leq 2, \quad |\theta| \leq 2 - \alpha. \tag{4.8b}$$

Le us conclude this section with some remarks. In the limiting case $\alpha \to 2^-$ (and consequently $\theta \to 0$) the standard (3-point) scheme for the common diffusion equation is recovered. In fact we just obtain (2.16) and (2.17) from (4.7b) and (4.8b) with $\alpha = 2$ and $\theta = 0$. This means that *exclusively* for approximation of standard diffusion only jumps of one step to the right or one to the left or jumps of width zero occur. For all other values of α, *i.e.* $0 < \alpha < 2$ with $\alpha \neq 1$, arbitrarily large jumps occur with power-like decaying probability, as it turns out from the asymptotic analysis for the transfer coefficients, and is to be expected from the corresponding properties of the Lévy distributions outlined in Appendix B.

In fact, using the asymptotic formula in the binomial coefficients entering (4.7a) and (4.7b), one finds*

$$p_{\pm k} \sim |c_\pm| \frac{\tau}{h^\alpha} \Gamma(\alpha+1) \frac{|\sin(\pi\,\alpha)|}{\pi} k^{-(\alpha+1)}, \quad k \to \infty. \tag{4.9}$$

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* The proof easily follows from the use of the reflection formula for the gamma function and Stirling's asymptotics. In fact, for all positive non-integer α

$$\left|\binom{\alpha}{k}\right| = \left|\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)}\right| = \Gamma(\alpha+1)\,\frac{|\sin(\pi\alpha)|}{\pi}\,\frac{|\Gamma(k-\alpha)|}{\Gamma(k+1)},$$

hence

$$\left|\binom{\alpha}{k}\right| \sim \Gamma(\alpha+1)\,\frac{|\sin(\pi\alpha)|}{\pi}\,k^{-(\alpha+1)} \quad \text{as} \quad k \to \infty.$$

A consequence is that, if $\alpha > 0$, the binomial series

$$\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k$$

converges absolutely and uniformly for $|z| \leq 1$, where it represents the function $(1-z)^\alpha$.

The result (4.9) thus provides the discrete counterpart of the expected asymptotic behaviour for the long power-law tails of the stable distributions when $0 < \alpha < 2$, see (B.11).

As pointed out in Appendix B, the only exceptions to the power-law tails occur in the extremal cases

$$(a) \quad 0 < \alpha < 1\,, \theta = \pm\alpha\,, \quad \text{or} \quad (b) \quad 1 < \alpha \le 2\,, \theta = \pm(2-\alpha)\,,$$

where the continuous stable densities can exhibit exponential tails. It is thus worthwhile and illuminating to look closely into the structure of the random walks in the extremal cases, omitting the (symmetric) Gauss case $\alpha = 2$.

In the cases (a) we expect one-sided stable densities restricted to $\mathbb{R}_0^+$ if $\theta = -\alpha$, or to $\mathbb{R}_0^-$ if $\theta = +\alpha$.

As an example we consider in detail $\theta = -\alpha$. In this case we have from (3.21a) $c_+ = 1$, $c_- = 0$, hence from (4.7a) and (4.8a)

$$\begin{cases} p_0 = 1 - \dfrac{\tau}{h^\alpha}\,, \\ p_{-k} = 0\,, \quad k = 1, 2, 3, \ldots \\ p_{+k} = \dfrac{\tau}{h^\alpha} \left| \dbinom{\alpha}{k} \right|\,, \quad k = 1, 2, 3, \ldots\,, \end{cases} \tag{4.10a}$$

with

$$0 < \frac{\tau}{h^\alpha} \le 1\,. \tag{4.11a}$$

This means no transition in negative direction, and thus the random walk is one-sided, the particle can only remain in its place or jump in positive direction, in analogy to the continuous process.

We note that for $\alpha = 1/2$ and $\theta = -1/2$ the numerical scheme can be checked in providing the numerical approximation of the one-sided stable *Lévy* distribution, see (B.12) *i.e.*

$$\mathcal{G}_c(x, t; 1/2, -1/2) = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\, \dfrac{t}{x^{3/2}}\, \mathrm{e}^{-t^2/(4x)}\,, \; x > 0\,, \\ 0\,, \qquad x \le 0\,. \end{cases} \tag{4.12}$$

In the cases (b) we expect two-sided stable densities, which if $1 < \alpha < 2$ exhibit both an algebraic and an exponential tail. For $\theta = +(2-\alpha)$ the exponential tail is at left, *i.e.* as $x \to -\infty$, so the maximum of the density is located in $\mathbb{R}^-$. For $\theta = -(2-\alpha)$ we have the reverse situation, *i.e.* the exponential tail is at right, *i.e.* as $x \to +\infty$, so the maximum of the density is located in $\mathbb{R}^+$.

As an example we consider in detail $\theta = 2 - \alpha$. In this case we have from (3.21b) $c_+ = -1$, $c_- = 0$, hence from (4.7b) and (4.8b)

$$\begin{cases} p_0 = 1 - \dfrac{\tau}{h^\alpha}\,\alpha\,, \\ p_{-1} = \dfrac{\tau}{h^\alpha}\,, \\ p_{-k} = 0\,, \quad k = 2, 3, \dots \\ p_{+k} = \dfrac{\tau}{h^\alpha} \left| \dbinom{\alpha}{k+1} \right|\,, \quad k = 1, 2, 3, \dots\,, \end{cases} \tag{4.10b}$$

with

$$0 < \frac{\tau}{h^\alpha} \le \frac{1}{\alpha}\,. \tag{4.11b}$$

This means that the random walk is not completely one-sided, the particle is mostly restricted to staying in place or jumping in positive direction, in analogy to the continuous process. At each time-step there is only a fraction p_{-1} of probability of jumping one step h to the left. In the limit $h \to 0$ this leads to some kind of (exponential) thin tail at the left. This again stresses the importance of the shifted definition of ${}_hD_\theta^\alpha$ for $1 < \alpha \le 2$.

5. Proof of Convergence

Let us briefly show that our random walk models indeed "converge" to the corresponding continuous processes.

Let us first consider the *generating functions*

$$\tilde{p}(z) := \sum_{k=-\infty}^{+\infty} p_k\, z^k\,, \quad \tilde{y}(z, t_n) := \sum_{j=-\infty}^{+\infty} y_j(t_n)\, z^j\,, \quad |z| = 1\,, \tag{5.1}$$

(the two series are absolutely and uniformly convergent on the periphery $|z| = 1$ of the unit circle) for the transition probabilities p_k and the sojourn probabilities $y_k(t_n)$. Noting that (2.13) describes a discrete convolution, we easily recognize by the discrete convolution theorem that

$$\tilde{y}(z, t_n) = \tilde{y}(z, 0)\, [\tilde{p}(z)]^n\,, \quad n \in \mathbb{N}\,, \quad |z| = 1\,.$$

With the special choice $y_k(0) = \delta_{k\,0}$ (the discrete delta function meaning that the particle starts at the origin $x = 0$) we have $\tilde{y}(z, 0) \equiv 1$, hence

$$\tilde{y}(z, t_n) = [\tilde{p}(z)]^n\,, \quad n \in \mathbb{N}\,, \quad |z| = 1\,. \tag{5.2}$$

Remembering that our random walk occurs on the grid $x_j = j\,h\,,\ j \in \mathbf{Z}$ (with $h > 0$), the change to the *characteristic functions* of our discrete probability distributions is accomplished by putting

$$z = \mathrm{e}^{i\,\kappa\,h}\,, \quad \kappa \in \mathbb{R}\,. \tag{5.3}$$

Now, referring to the distinction (a) and (b) of Section 4 and to formulas (4.7a) and (4.7b) for the transition probabilities, we see (by aid of the binomial theorem) that with the coefficients c_+ and c_- given in (3.11), and the scaling parameter

$$\mu := \frac{\tau}{h^\alpha}\,, \tag{5.4}$$

we have :

in case (a) $\quad 0 < \alpha < 1\,, \quad |\theta| \le \alpha\,,$

$$\tilde{p}(z) = 1 - \mu\,\left[c_+\,(1-z)^\alpha + c_-\,(1-z^{-1})^\alpha\right]\,, \tag{5.5a}$$

in case (b) $\quad 1 < \alpha \le 2\,, \quad |\theta| \le 2-\alpha\,,$

$$\tilde{p}(z) = 1 - \mu\,\left[c_+\,z^{-1}\,(1-z)^\alpha + c_-\,z\,(1-z^{-1})^\alpha\right]\,. \tag{5.5b}$$

Fixing now a positive real number t and letting the time step $\tau \to 0$ so that $n = t/\tau \to \infty$ through the positive integers, we have $t = t_n$ and

$$\tilde{y}(z,t) = [\tilde{p}(z)]^{t/\tau}\,. \tag{5.6}$$

Changing to characteristic functions via (5.3) while fixing the scaling parameter μ as a a positive number subject to the restrictions (4.8a) or (4.8b) and writing $\tau = \mu\,h^\alpha$, see (5.4), we obtain for the characteristic function of the sojourn probabilities at time t the expression

$$\tilde{y}\left(\mathrm{e}^{i\,\kappa\,h}, t\right) = \left[\tilde{p}\left(\mathrm{e}^{i\,\kappa\,h}\right)\right]^{t/(\mu\,h^\alpha)}\,. \tag{5.7}$$

By sending $h \to 0$ the grid becomes infinitely fine and also $\tau \to 0$, and remembering the initial condition as the discrete delta function, we expect that (5.7) will tend to the Fourier transform (3.5) of the corresponding Lévy-Feller process.

It is now an exercise in complex analysis (a not quite trivial one) to verify that indeed, as desired, the relation

$$\mathop{lim}_{h\to 0}\,\tilde{y}\left(\mathrm{e}^{i\,\kappa\,h}, t\right) = \exp\left\{-t\,|\kappa|^\alpha\,\mathrm{e}^{\,i(\mathrm{sign}\,\kappa)\,\theta\,\pi/2}\right\}\,, \tag{5.8}$$

holds in both cases (a) and (b). Invoking Theorem 3.6.1 of Lukacs [24], we obtain convergence of the cumulative distribution function of the random walk model to the cumulative distribution function of the continuous process whose characteristic function is the R.H.S. of (5.8), taking into account that the latter cumulative distribution function is everywhere continuous. The proof is so completed.

In a forthcoming paper we will present a more detailed version of the proof, furthermore investigation of the random walk model as a difference scheme in the sense of numerical analysis with regards to degrees of stability and speed of convergence (order of accuracy for small h).

Appendix A: Fourier Transform and Pseudo-differential Operators

If $\phi(x)$ denotes a sufficiently well-behaved function * in $\mathbb{R}$, the Fourier transform and its inverse read

$$\begin{cases} \hat{\phi}(\kappa) = \mathcal{F}\left[\phi(x)\right] = \displaystyle\int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x}\, \phi(x)\, dx\,, \\ \phi(x) = \mathcal{F}^{-1}\left[\hat{\phi}(\kappa)\right] = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} \mathrm{e}^{-i\kappa x}\, \hat{\phi}(\kappa)\, d\kappa\,, \end{cases} \qquad (A.1)$$

where $\kappa \in \mathbb{R}$. Let us introduce the notation

$$\phi(x) \div \hat{\phi}(\kappa)\,, \quad x\,,\, \kappa \in \mathbb{R}\,. \qquad (A.2)$$

Let us recall the *convolution theorem*

$$\phi_1(x) \star \phi_2(x) := \int_{-\infty}^{+\infty} \phi_1(\xi)\, \phi_2(x-\xi)\, d\xi \;\div\; \hat{\phi}_1(\kappa)\, \hat{\phi}_2(\kappa)\,, \qquad (A.3)$$

and the Fourier representation of the derivatives, namely

$$\frac{d^n}{dx^n}\, \phi(x) \div (-i\kappa)^n\, \hat{\phi}(\kappa)\,, \quad x\,,\, \kappa \in \mathbb{R}\,, \quad n \in \mathbb{N}\,. \qquad (A.4)$$

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* An infinitely often differentiable function $\phi(x)$ which has bounded support or decreases towards zero faster than any negative power of $|x|$ as $|x| \to \infty$ is more than good enough. The set of such functions is dense in the space $L_1(\mathbb{R})$ of Lebesgue integrable functions, the space naturally appropriate for densities of probabilities or diffusing substances, and the solution operator of our Cauchy problem can be extended to this space.

We easily recognize

$$\frac{d^2}{dx^2}\,\phi(x) \div -|\kappa|^2\,\hat{\phi}(\kappa)\,, \quad x\,,\,\kappa \in \mathbb{R}\,, \qquad (A.5)$$

taking into account that $\kappa^2 = |\kappa|^2$.

For further use, we note that in analogy to (A.4) we can define a generic pseudo-differential operator A (of which the n-th derivative operator is a special case) through its Fourier representation, namely

$$A\,\phi(x) \div \hat{A}(\kappa)\,\hat{\phi}(\kappa)\,. \qquad (A.6)$$

According to a usual terminology, the term $\hat{A}(\kappa)$ preceeding $\hat{\phi}(\kappa)$ is referred to as the symbol of the pseudo-differential operator A; so, because of (A.5), the symbol of the 2-nd derivative operator turns out to be $-|\kappa|^2$.

Here we do not want to expand on the functional-analytic aspects for which we refer the reader to treatises on semigroup theory and pseudo-differential operators, *e.g.* Goldstein [12], Jacob [17].

Appendix B: The Stable Probability Distributions

The stable distributions are a fascinating and fruitful area of research in probability theory; furthermore, nowadays, they provide valuable models in physics, astronomy, economics, and communication theory.

The general class of stable distributions was introduced and given this name by the French mathematician Paul Lévy in the early 1920's, see Lévy [21-22]. The inspiration for Lévy was the desire to generalize the celebrated *Central Limit Theorem*, according to which any probability distribution with finite variance belongs to the domain of attraction of the Gaussian distribution.

Formerly, the topic attracted only moderate attention from the leading experts, though there were also enthusiasts, of whom the Russian mathematician Alexander Yakovlevich Khintchine should be mentioned first of all. The concept of stable distributions took full shape in 1937 with the appearance of Lévy's monograph [23], soon followed by Khintchine's monograph [19].

The theory and properties of stable distributions are discussed in some classical books on probability theory including Gnedenko & Kolmogorov [11], Lukacs [24], Feller [9], Breiman [4], Chung [6] and Laha & Rohatgi [20]. Also treatises on fractals devote particular attention to stable distributions in view of their properties of scale invariance, see *e.g.* Mandelbrot [25] and Takayasu [34].

Only recently, monographs devoted solely to stable distributions and related stochastic processes have appeared, *i.e.* Zolotarev [36] Janicki & Weron [18], and Samorodnitsky & Taqqu [32].

Stable distributions have three *exclusive* properties, which can be briefly summarized by stating that they 1) are *invariant under addition*, 2) possess their *own domain of attraction*, and 3) admit a *canonic characteristic function.*

Let us now illustrate the above properties which, providing necessary and sufficient conditions, can be assumed as equivalent definitions for a stable distribution. We recall the basic results without proof, essentially referring to [32].

A random variable X is said to have a stable distribution $P(x) = \text{Prob}\,\{X \le x\}$ if for any $n \ge 2$, there is a positive number c_n and a real number d_n such that

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} c_n\, X + d_n\,, \qquad (B.1)$$

where $X_1, X_2, \ldots X_n$ denote mutually independent random variables with common distribution $P(x)$ with X. Here the notation $\stackrel{d}{=}$ denotes equality in distribution, *i.e.* means that the random variables on both sides have the same probability distribution.

When mutually independent random variables have a common distribution [shared with a given random variable X], we also refer to them as independent, identically distributed (i.i.d) random variables [independent copies of X]. In general, the sum of i.i.d. random variables becomes a random variable with a distribution of different form. However, for independent random variables with a common *stable* distribution, the sum obeys to a distribution of the same type, which differs from the original one only for a scaling (c_n) and possibly for a shift (d_n). When in (B.1) the $d_n = 0$ the distribution is called *strictly stable.*

It is known, see Feller [9], that the norming constants in (B.1) are of the form

$$c_n = n^{1/\alpha} \quad \text{with} \quad 0 < \alpha \le 2\,. \qquad (B.2)$$

The parameter α is called the *characteristic exponent* or the *index of stability* of the stable distribution.

We agree to use the notation $X \sim P_\alpha(x)$ to denote that the random variable X has a stable probability distribution with characteristic exponent α. We simply refer to $P(x)$, $p(x) := dP/dx$ (probability density function = *pdf*) and X as *α-stable distribution, density, random variable*, respectively.

A stable distribution is called *symmetric* if the random variable $-X$ has the same distribution. Of course, a *symmetric* stable distribution is necessarily *strictly stable*.

Noteworthy examples of symmetric stable distributions are provided by the Gaussian (or normal) law (with $\alpha = 2$) and by the Cauchy-Lorentz law ($\alpha = 1$). The corresponding *pdf* are known to be

$$p_G(x;\sigma) := \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-x^2/(2\sigma^2)}\,, \quad x \in \mathbb{R}\,, \qquad (B.3)$$

where σ^2 denotes the variance, and

$$p_C(x;\gamma) := \frac{1}{\pi}\,\frac{\gamma}{x^2+\gamma^2}\,, \quad x \in \mathbb{R}\,, \qquad (B.4)$$

where γ denotes the semi-interquartile range. We note that for the *Gauss pdf* the absolute moments of any order $\nu \geq 0$ are finite, whereas for the *Cauchy pdf* only the absolute moments of order $0 \leq \nu < 1$ are finite, and in particular the variance is infinite.

Another (equivalent) definition states that stable distributions are the only distributions that can be obtained as limits of normalized sums of i.i.d. random variables. A random variable X is said to have a *domain of attraction,i.e.* if there is a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers $\{\gamma_n\}$ and real numbers $\{\delta_n\}$, such that

$$\frac{Y_1 + Y_2 + \ldots Y_n}{\gamma_n} + \delta_n \stackrel{d}{\Rightarrow} X\,. \qquad (B.5)$$

The notation $\stackrel{d}{\Rightarrow}$ denotes convergence in distribution.

If the random variable X has a stable distribution and all Y_i are taken to be independent and distributed like X, then clearly (B.1) implies (B.5), and so trivially every random variable with a stable probability distribution has a domain of attraction. The converse is also true, namely that every random variable with a domain of attraction has a stable probability distribution, see Gnedenko and Kolmogorov [11]. Therefore we can alternatively state that *a random variable X is said to have a stable distribution if it has a domain of attraction.*

When X is Gaussian and the Y_is are i.i.d. with finite variance, then (B.5) is the statement of the ordinary *Central Limit Theorem*.

Another definition specifies the *canonic form* that the *characteristic function* (cf) of a stable distribution of index α must have. Recalling that the cf is the Fourier transform of the *pdf*, we use the notation $\hat{p}_\alpha(\kappa) := \langle \exp(i\kappa X)\rangle \div p_\alpha(x)$.

We note that the cf for the most popular stable distributions, the Gauss and Cauchy *pdf*, are known to be

$$\widehat{p}_G(\kappa;\sigma) = \mathrm{e}^{-(\sigma^2/2)\,|\kappa|^2}\,, \qquad (B.6)$$

$$\widehat{p}_c(\kappa;\gamma) = \mathrm{e}^{-\gamma|\kappa|}\,. \qquad (B.7)$$

If we neglect the inessential parameters related to scale transformations (and pure translation) we can adopt the simplified canonic form used by Feller, [7], [9], and Takayasu [34] for strictly stable distributions, which reads in our notation,

$$\hat{p}_\alpha(\kappa;\theta) := \exp\left\{-|\kappa|^\alpha\, \mathrm{e}^{\,i(\mathrm{sign}\,\kappa)\,\theta\,\pi/2}\right\}\,. \qquad (B.8)$$

Here θ is the *skewness* parameter and its domain is restricted to the following region (depending on α)

$$\begin{cases} |\theta| \le \alpha\,, & \text{if } 0<\alpha<1\,,\\ |\theta| \le 2-\alpha\,, & \text{if } 1<\alpha<2\,,\\ \theta = 0\,, & \text{if } \alpha = 1\,,\,2\,. \end{cases} \qquad (B.9)$$

We recognize that $p_\alpha(x,\theta) = p_\alpha(-x,-\theta)$, so the *symmetric* stable distributions are obtained if and only if $\theta = 0$. Particular cases of the canonic form are provided by the Gaussian distribution (B.6) with $\sigma^2 = 2$, and the Cauchy distribution (B.7) with $\gamma = 1$.

For all the other stable distributions Feller [7], [9] has obtained from (B.8) the following representations in terms of convergent power series valid for $x > 0$:

(a) $0 < \alpha < 1$ (negative powers),

$$p_\alpha(x;\theta) = \frac{1}{\pi\, x}\sum_{n=1}^{\infty}(-x^{-\alpha})^n\,\frac{\Gamma(n\alpha+1)}{n!}\,\sin\left[\frac{n\pi}{2}(\theta-\alpha)\right]\,, \qquad (B.10a)$$

(b) $1 < \alpha \leq 2$ (positive powers),

$$p_\alpha(x;\theta) = \frac{1}{\pi\, x} \sum_{n=1}^{\infty} (-x)^n \, \frac{\Gamma(n/\alpha + 1)}{n!} \sin\left[\frac{n\pi}{2\alpha}(\theta - \alpha)\right] . \qquad (B.10b)$$

The values for $x < 0$ can be obtained from (B.10) using the identity $p_\alpha(-x;\theta) = p_\alpha(x;-\theta)$, $x > 0$.

It has be shown, see *e.g.* Bergström [2], Chao Chung-Jeh [5], that the two series in (B.10) provide also the asymptotic (divergent) expansions to the stable *pdf* with the ranges of α interchanged from those of convergence, that is the series at the R.H.S of (B.10a) is asymptotic for the L.H.S of (B.10b) as $x \to \infty$, and the series at the R.H.S. of (B.10b) is asymptotic for the L.H.S of (B.10a) as $x \to 0$.

Stable distributions with extremal values of the skewness parameter are called *extremal*. One can prove that all the extremal stable distributions with $0 < \alpha < 1$ are one-sided, the support being $\mathbb{R}_0^+$ if $\theta = -\alpha$, and $\mathbb{R}_0^-$ if $\theta = +\alpha$.

For $0 < \alpha < 2$ the stable distributions exhibit heavy tails in such a way that their absolute moment of order ν is finite only if $\nu < \alpha$. In fact one can show that for non-Gaussian, not extremal, stable distributions the asymptotic decay of the tails is

$$p_\alpha(x;\theta) = O\left(|x|^{-(\alpha+1)}\right) , \quad x \to \pm\infty . \qquad (B.11)$$

For the extremal distributions this is valid only for one tail, the other being of exponential order. For $0 < \alpha < 1$ we have one-sided distributions which exhibit an exponential left tail (as $x \to 0^+$) if $\theta = -\alpha$, or an exponential right tail (as $x \to 0^-$) if $\theta = +\alpha$. For $1 < \alpha < 2$ the extremal distributions are two-sided and exhibit an exponential left tail (as $x \to -\infty$) if $\theta = +(2-\alpha)$, or an exponential right tail (as $x \to +\infty$) if $\theta = -(2-\alpha)$.

Consequently, the Gaussian distribution is the unique stable distribution with finite variance. Furthermore, when $\alpha \leq 1$, the first absolute moment $\langle |X| \rangle$ is infinite as well, so we need to use the median to characterize the expected value.

However, there is a fundamental property shared by all the stable distributions that we like to point out: for any α the corresponding stable *pdf* is *unimodal* and indeed *bell-shaped*, *i.e.* its n-th derivative has exactly n zeros, see Gawronski [10].

A general representation of all stable distributions in terms of H functions has been only recently achieved by Schneider [33]. However, in addition to the Gauss and Cauchy distributions there are other two noteworthy examples of stable distributions admitting simple explicit expressions; they are the one-sided distributions (with support in $\mathbb{R}_0^+$),

$$p_{1/2}(x;-1/2) = \frac{1}{2\sqrt{\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/(4x)}\,, \quad x \geq 0\,, \tag{B.12}$$

$$p_{1/3}(x;-1/3) = 3^{-1/3}\, x^{-4/3}\, \mathrm{Ai}\left[(3x)^{-1/3}\right]\,, \quad x \geq 0\,, \tag{B.13}$$

where Ai denotes the Airy function, see *e.g.* Abramowitz and Stegun [1].

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