Perspectives in Fractional Calculus: FDEs in Non-Euclidean world

A possible direction of extension of research of fractional calculus can be that of fractionalizing the non-Euclidean world.
First of all the so-called hyperbolic Brownian motion, that is a Brownian motion on the Poincare' half-plane H_2^+ (or more generally H_n^+) is a diffusion whose probability density $p(\eta, t)$ (where η is the hyperbolic distance from the origin of H_n^+) satisfy the p.d.e.

$$\frac{\partial p}{\partial t} = \frac{1}{\sinh^{n-1}\eta}\frac{\partial}{\partial \eta}(\sinh^{n-1}\eta)p, \quad n \geq 2, \eta > 0, t > 0.$$

If $\partial_t p$ is replaced by the fractional derivative $\partial_t^\nu p$, one obtain a first "naive" fractionalization. If $\nu = 1/2$ it was shown in Lao and Orsingher (2007) that $p(\cdot)$ is the density of the composition of an hyperbolic Brownian motion with a reflecting standard Brownian motion. In this framework other cases can be investigated in much detail.

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Hyperbolic telegraph equations have also been fractionalized (see for example D'Ovidio et al. (2014)) where equations of the type

$$\frac{\partial^{2\nu} p}{\partial t^{2\nu}} + 2\lambda \frac{\partial^{\nu} p}{\partial t^{\nu}} = \frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta} (\sinh^{n-1}\eta) p, \quad n \geq 2, \eta > 0, t > 0,$$

have been considered. Also in this case the fractional pendant is by far still not thoroughly analyzed.

Literature:

M.D'Ovidio, E. Orsingher, B.Toaldo, Fractional telegraph-type equations and hyperbolic Brownian motion, Statistics and Probability Letters, 89: 131 - 137, 2014

L.Lao, E. Orsingher, Hyperbolic and fractional hyperbolic Brownian motion with some applications. Stochastics, Vol 79, 6 pag.505-522, 2007

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