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FRACTIONAL CALCULUS AND SPECIAL FUNCTIONS

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Abstract

The aim of these introductory lectures is to provide the reader with the essentials of the fractional calculus according to different approaches that can be useful for our applications in the theory of probability and stochastic processes. We discuss the linear operators of fractional integration and fractional differentiation, which were introduced in pioneering works by Abel, Liouville, Riemann, Weyl, Marchaud, M. Riesz, Feller and Caputo. Particular attention is devoted to the techniques of Fourier and Laplace transforms for treating these operators in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor. Furthermore, we discuss the approach based on limit of difference quotients, formerly introduced by Grünwald and Letnikov, which provides a discrete access to the fractional calculus. Such approach is very useful for actual numerical computation and is complementary to the previous integral approaches, which provide the continuous access to the fractional calculus. Finally, we give some information on the higher transcendental functions of the Mittag-Leffler and Wright type which, together with the most common Eulerian functions, turn out to play a fundamental role in the theory and applications of the fractional calculus. We refrain for treating the more general functions of the Fox type (H functions), referring the interested reader to specialized papers and books.

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A. Historical Notes and Introduction to Fractional Calculus

The development of the fractional calculus

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term *fractional* is a misnomer, but it is retained following the prevailing use.

The fractional calculus may be considered an *old* and yet *novel* topic. It is an *old* topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. In fact the idea of generalizing the notion of derivative to non integer order, in particular to the order 1/2, is contained in the correspondence of Leibniz with Bernoulli, L'Hôpital and Wallis. Euler took the first step by observing that the result of the evaluation of the derivative of the power function has a a meaning for non-integer order thanks to his Gamma function.

A list of mathematicians, who have provided important contributions up to the middle of the 20-th century, includes P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1837), B. Riemann (1847), H. Holmgren (1865-67), A.K. Grünwald (1867-1872), A.V. Letnikov (1868-1872), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892-1912), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924-1936), A. Zygmund (1935-1945), E.R. Love (1938-1996), A. Erdélyi (1939-1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949), W. Feller (1952).

However, it may be considered a *novel* topic as well, since only from less than thirty years ago it has been object of specialized conferences and treatises. The merit is due to B. Ross for organizing the *First Conference on Fractional Calculus and its Applications* at the University of New Haven in June 1974 and editeding the proceedings [112]. For the first monograph the merit is ascribed to K.B. Oldham and J. Spanier [105], who, after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974.

Nowadays, to our knowledge, the list of texts in book form with a title explicitly devoted to fractional calculus (and its applications) includes around ten titles, namely Oldham & Spanier (1974) [105] McBride (1979) [93], Samko, Kilbas & Marichev (1987-1993) [117], Nishimoto (1991) [104], Miller & Ross (1993) [97], Kiryakova (1994) [68], Rubin (1996) [113], Podlubny (1999) [107], and Kilbas, Strivastava & Trujillo (2006) [67]. Furthermore, we recall the attention to the treatises by Davis (1936) [30], Erdélyi (1953-1954) [37], Gel'fand & Shilov (1959-1964) [43], Djrbashian (or Dzherbashian) [31, 32], Caputo [18], Babenko [5], Gorenflo & Vessella [52], West, Bologna & Grigolini (2003) [127], Zaslavsky (2005) [139], Magin (2006) [78], which contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus.

For more details on the historical development of the fractional calculus we refer the interested reader to Ross' bibliography in [105] and to the historical notes generally available in the above quoted texts.

In recent years considerable interest in fractional calculus has been stimulated by the applications that it finds in different fields of science, including numerical analysis, economics and finance, engineering, physics, biology, etc.

For economics and finance we quote the collection of articles on the topic of *Fractional Differencing and Long Memory Processes*, edited by Baillie & King (1996), appeared as a special issue in the Journal of Econometrics [6]. For engineering and physics we mention the book edited by Carpinteri and Mainardi (1997) [28], entitled *Fractals and Fractional Calculus in Continuum Mechanics*, which contains lecture notes of a CISM Course devoted to some applications of related techniques in mechanics, and the book edited by Hilfer (2000) [57], entitled *Applications of Fractional Calculus in Physics*, which provides an introduction to fractional calculus for physicists, and collects review articles written by some of the leading experts. In the above books we recommend the introductory surveys on fractional calculus by Gorenflo & Mainardi [48] and by Butzer & Westphal [16], respectively

Besides to some books containing proceedings of international conferences and workshops on related topics, see *e.g.* [94], [103], [114], [115], we mention regular journals devoted to fractional calculus, *i.e. Journal of Fractional Calculus* (Descartes Press, Tokyo), started in 1992, with Editor-in-Chief Prof. Nishimoto and *Fractional Calculus and Applied Analysis* from 1998, with Editor-in-Chief Prof. Kiryakova (Diogenes Press, Sofia). For information on this journal, please visit the WEB site `www.diogenes.bg/fcaa`. Furthermore WEB sites devoted to fractional calculus have been appointed, of which we call attention to `www.fracalmo.org` whose name is originated by FRActional CALculus MOdelling, and the related WEB links.

The approaches to fractional calculus

There are different approaches to the fractional calculus which, not being all equivalent, have lead to a certain degree of confusion and several misunderstandings in the literature. Probably for this the fractional calculus is in some way the "black sheep" of the analysis. In spite of the numerous eminent mathematicians who have worked on it, still now the fractional calculus is object of so many prejudices. In these review lectures we essentially consider and develop two different approaches to the fractional calculus in the framework of the real analysis: the *continuous* one, based integral operators and the *discrete* one, based on infinite series of finite differences with increments tendig to zero. Both approaches turn out to be useful in treating our generalized diffusion processes in the theory of probability and stochastic processes.

For the continuous approach it is customary to generically refer to the *Riemann-Liouville fractional calculus*, following a terminology introduced by Holmgren (1865) [59]. We prefer to distinguish three kinds of fractional calculus: *Liouville-Weyl fractional calculus*, *Riesz-Feller fractional calculus*, and *Abel-Riemann fractional calculus*, which, concerning three different types of integral operators acting on unbounded domains, are of major interest for us. We shall devote the next three sections, B, C and D, to the above kinds of fractional calculus, respectively. However, as an introduction, we shall hereafter consider fractional integrals and derivatives based on integral operators acting on bounded domains.

For the discrete approach one usually refers to the *Grünwald-Letnikov fractional calculus*. We shall devote section E to this approach. The finite difference schemes treated in this framework turn out to be useful in the interpretation of our fractional diffusion processes by means of discrete random-walk models. In this respect we may refer the reader to our papers already published, see *e.g.* [49, 50, 51], looking forward to the last part of our planned lecture notes.

Finally, sections F and G are devoted respectively to the special (higher transcendental) functions of the Mittag-Leffler and Wright type, which play a fundamental role in our applications of the fractional calculus. We refrain for treating the more general Fox H functions, referring the interested reader to specialized papers and books.

We use the standard notations $\mathbb{N}$, $\mathbf{Z}$, $\mathbb{R}$, $\mathbb{C}$ to denote the sets of natural, integer, real and complex numbers, respectively; furthermore, $\mathbb{R}^+$ and $\mathbb{R}^+_0$ denote the sets of positive real numbers and of non-negative real numbers, respectively. Let us remark that, wanting our lectures to be accessible to various kinds of people working in applications (*e.g.* physicists, chemists, theoretical biologists, economists, engineers) here we, deliberately and consciously as far as possible, avoid the language of functional analysis. We thus use vague phrases like "for a sufficiently well behaved function" instead of constructing a stage of precisely defined spaces of admissible functions. We pay particular attention to the techniques of integral transforms: we limit ourselves to Fourier, Laplace and Mellin transforms for which notations and main properties will be recalled as soon as they become necessary. We make formal use of generalized functions related to the Dirac "delta function" in the typical way suitable for applications in physics and engineering, without adopting the language of distributions. We kindly ask specialists of these fields of pure mathematics to forgive us. Our notes are written in a way that makes it easy to fill in details of precision which in their opinion might be lacking.

We now present an introductory survey of the Riemann-Liouville fractional calculus. We do hope that the reader can immediately get a preliminary idea on fractional calculus, including notations and subtleties, before reading the subsequent, more technical, sections B, C, D.

Introduction to the Riemann-Liouville fractional calculus

As it is customary, let us take as our starting point for the development of the so called Riemann-Liouville fractional calculus the repeated integral

$$I_{a+}^{n}\,\phi(x) := \int_a^x \int_a^{x_{n-1}} \ldots \int_a^{x_1} \phi(x_0)\,dx_0\,\ldots\,dx_{n-1}\,,\quad a \le x < b\,,\; n \in \mathbb{N}\,, \qquad (A.1)$$

where $a > -\infty$ and $b \le +\infty$. The function $\phi(x)$ is assumed to be well behaved; for this it suffices that $\phi(x)$ is locally integrable in the interval $[a,b)$, meaning in particular that a possible singular behaviour at $x = a$ does not destroy integrability. It is well known that the above formula provides an $n-$fold primitive $\phi_n(x)$ of $\phi(x)$, precisely that primitive which vanishes at $x = a$ jointly with its derivatives of order $1, 2, \ldots n-1$.

We can re-write this n-fold repeated integral by a covolution-type formula (often attributed to Cauchy) as, see **Problem B.1**,

$$I_{a+}^{n}\,\phi(x) = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1}\,\phi(\xi)\,d\xi\,,\quad a \le x < b\,. \qquad (A.2)$$

In a natural way we are now led to extend the formula (A.2) from positive integer values of the index n to arbitrary positive values α, thereby using the relation $(n-1)! = \Gamma(n)$. So, using the Gamma function, we define the *fractional integral of order* α as

$$I_{a+}^{\alpha}\,\phi(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi\,,\quad a < x < b\,,\quad \alpha > 0\,. \qquad (A.3)$$

We remark that the values $I_{a+}^{n}\,\phi(x)$ with $n \in \mathbb{N}$ are always finite for $a \le x < b$, but the values $I_{a+}^{\alpha}\,\phi(x)$ for $\alpha > 0$ are assumed to be finite for $a < x < b$, whereas, as we shall see later, it may happen that the limit (if it exists) of $I_{a+}^{\alpha}\phi(x)$ for $x \to a^+$, that we denote by $I_{a+}^{\alpha}\,\phi(a^+)$, is infinite.

Without loss of generality, it may be convenient to set $a = 0$. We agree to refer to the fractional integrals I_{0+}^{α} as to the *Abel-Riemann fractional integrals*, honouring both the authors who first treated similar integrals[1].

For I_{0+} we use the special and simplified notation J^{α} in agreement with the notation introduced by Gorenflo & Vessella (1991) [52] and then followed in all our papers in the subject. We shall return to the fractional integrals J^{α} in Section D providing a sufficiently exhaustive treatment of the related fractional calculus.

[1]Historically, fractional integrals of type (A.3) were first investigated in papers by Abel (1823) (1826) [1, 2] and by Riemann (1876) [110]. In fact, Abel, when he introduced his integral equation, named after him, to treat the problem of the tautochrone, was able to find the solution inverting a fractional integral of type (A.3). The contribution by Riemann is supposed to be independent from Abel and inspired by previous works by Liouville (see later): it was written in January 1847 when he was still a student, but published in 1876, ten years after his death.

A dual form of the integrals (A.2) is

$$I_{b-}^{n}\,\phi(x) = \frac{1}{(n-1)!}\int_{x}^{b}(\xi - x)^{n-1}\,\phi(\xi)\,d\xi\,,\quad a < x \le b\,, \tag{A.4}$$

where we assume $\phi(x)$ to be sufficiently well behaved in $-\infty \le a < x < b < +\infty$. Now it suffices that $\phi(x)$ is locally integrable in $(a,b]$.

Extending (A.4) from the positive inegers n to $\alpha > 0$ we obtain the dual form of the fractional integral (A.3), *i.e.*

$$I_{b-}^{\alpha}\,\phi(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(\xi - x)^{\alpha-1}\,\phi(\xi)\,d\xi\,,\quad a < x \le b \quad \alpha > 0\,, \tag{A.5}$$

Now it may happen that the limit (if it exists) of $I_{b-}^{\alpha}\phi(x)$ for $x \to b^{-}$, that we denote by $I_{b-}^{\alpha}\,\phi(b^{-})$, is infinite.

We refer to the fractional integrals I_{a+}^{α} and I_{b-}^{α} as *progressive* or right-handed and *regressive* or left-handed, respectively.

Let us point out the fundamental property of the fractional integrals, namely the additive index law (*semi-group property*) according to which

$$I_{a+}^{\alpha}\,I_{a+}^{\beta} = I_{a+}^{\alpha+\beta}\,,\quad I_{b-}^{\alpha}\,I_{b-}^{\beta} = I_{b-}^{\alpha+\beta}\,,\quad \alpha\,,\,\beta \ge 0\,, \tag{A.6}$$

where, for complementation, we have defined $I_{a+}^{0} = I_{b-}^{0} := \mathbb{I}$ (Identity operator) which means $I_{a+}^{0}\,\phi(x) = I_{b-}^{0}\,\phi(x) = \phi(x)$. The proof of (A.6) is based on Dirichlet's formula concerning the change of the order of integration and the use of the Beta function in terms of Gamma function, see **Problem (A.2)**.

We note that the fractional integrals (A.3) and (A.5) contain a weakly singular kernel only when the order is less than one.

We can now introduce the concept of fractional derivative based on the fundamental property of the common derivative of integer order n

$$D^{n}\,\phi(x) = \frac{d^{n}}{dx^{n}}\,\phi(x) = \phi^{(n)}(x)\,,\quad a < x < b$$

to be the *left inverse operator* of the progressive repeated integral of the same order n, $I_{a+}^{n}\,\phi(x)$. In fact, it is straightforward to recognize that the derivative of any integer order $n = 0, 1, 2, \ldots$ satisfies the following composition rules with respect to the repeated integrals of the same order n, $I_{a+}^{n}\,\phi(x)$ and $I_{b-}^{n}\,\phi(x)$:

$$\begin{cases} D^{n}\,I_{a+}^{n}\,\phi(x) = \phi(x)\,, \\ \\ I_{a+}^{n}\,D^{n}\,\phi(x) = \phi(x) - \displaystyle\sum_{k=0}^{n-1}\frac{\phi^{(k)}(a^{+})}{k!}\,(x-a)^{k}\,, \end{cases} \quad a < x < b\,, \tag{A.7}$$

$$\begin{cases} D^n\, I^n_{b-}\, \phi(x) = (-1)^n \phi(x)\,, \\ \\ I^n_{b-}\, D^n\, \phi(x) = (-1)^n \left\{ \phi(x) - \displaystyle\sum_{k=0}^{n-1} (-1)^k\, \frac{\phi^{(k)}(b^-)}{k!}\, (b-x)^k \right\}\,, \end{cases} \quad a < x < b\,. \qquad (A.8)$$

In view of the above properties we can define the progressive/regressive fractional derivatives of order $\alpha > 0$ provided that we first introduce the positive integer m such that $m-1 < \alpha \le m$. Then we define the *progressive/regressive fractional derivative of order* α, $D^\alpha_{a+}/D^\alpha_{b-}$, as the *left inverse* of the corresponding fractional integral $I^\alpha_{a+}/I^\alpha_{b-}$. As a consequence of the previous formulas (A.2)-(A.8) the *fractional derivatives of order* α turn out to be defined as follows

$$\begin{cases} D^\alpha_{a+}\, \phi(x) := D^m\, I^{m-\alpha}_{a+}\, \phi(x)\,, \\ \\ D^\alpha_{b-}\, \phi(x) := (-1)^m\, D^m\, I^{m-\alpha}_{b-}\, \phi(x)\,, \end{cases} \quad a < x < b\,, \quad m-1 < \alpha \le m\,. \qquad (A.9)$$

In fact, for the progressive/regressive derivatives we get

$$\begin{cases} D^\alpha_{a+}\, I^\alpha_{a+} = D^m\, I^{m-\alpha}_{a+}\, I^\alpha_{a+} = D^m\, I^m_{a+} = \mathbb{I}\,, \\ \\ D^\alpha_{b-}\, I^\alpha_{b-} = (-1)^m\, D^m\, I^{m-\alpha}_{b-}\, I^\alpha_{b-} = (-1)^m\, D^m\, I^m_{b-} = \mathbb{I}\,. \end{cases} \qquad (A.10)$$

For complementation we also define $D^0_{a+} = D^0_{b-} := \mathbb{I}$ (Identity operator).

When the order is not integer $(m-1 < \alpha < m)$ the definitions (A.9) with (A.3), (A.5) yield the explicit expressions

$$D^\alpha_{a+}\, \phi(x) = \frac{1}{\Gamma(m-\alpha)}\, \frac{d^m}{dx^m} \int_a^x (x-\xi)^{m-\alpha-1}\, \phi(\xi)\, d\xi\,, \quad a < x < b\,, \qquad (A.11)$$

$$D^\alpha_{b-}\, \phi(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\, \frac{d^m}{dx^m} \int_x^b (\xi-x)^{m-\alpha-1}\, \phi(\xi)\, d\xi\,, \quad a < x < b\,. \qquad (A.12)$$

We stress the fact that the "proper" fractional derivatives (namely when α is non integer) are *non-local operators* being expressed by ordinary derivatives of convolution-type integrals with a weakly singular kernel, as it is evident from (A.11)-(A.12). Furthermore they do not obey necessarily the analogue of the "semi-group" property of the fractional integrals: in this respect the initial point $a \in \mathbb{R}$ (or the ending point $b \in \mathbb{R}$) plays a "disturbing" role. We also note that when the order α tends to both the end points of the interval $(m-1, m)$ we recover the common derivatives of integer order (unless of the sign for the regressive derivative).

B. The Liouville-Weyl Fractional Calculus

We can also define the fractional integrals over unbounded intervals and, as left inverses, the corresponding fractional derivatives. If the function $\phi(x)$ is locally integrable in $-\infty < x < b \le +\infty$, and behaves for $x \to -\infty$ in such away that the following integral exists in an appropriate sense, we can define the *Liouville fractional integral of order* α as

$$I_+^\alpha\,\phi(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \quad -\infty < x < b\,, \quad \alpha > 0\,. \qquad (B.1)$$

Analogously, if the function $\phi(x)$ is locally integrable in $-\infty \le a < x < +\infty$, and behaves well enough for $x \to +\infty$, we define the *Weyl fractional integral of order* α as

$$I_-^\alpha\,\phi(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}(\xi-x)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \quad a < x < +\infty\,, \quad \alpha > 0\,. \qquad (B.2)$$

Note the kernel $(\xi - x)^{\alpha-1}$ for (B.2). The names of Liouville and Weyl are here adopted for the fractional integrals (B.1), (B.2), respecively, following a standard terminology, see *e.g.* Butzer & Westphal (2000) [16], based on historical reasons[2].

We note that a sufficient condition for the integrals entering $I_\pm^\alpha$ in (B.1)-(B.2) to converge is that

$$\phi\,(x) = O\left(|x|^{-\alpha-\epsilon}\right)\,, \quad \epsilon > 0\,, \quad x \to \mp\infty\,,$$

respectively. Integrable functions satisfying these properties are sometimes referred to as functions of Liouville class (for $x \to -\infty$), and of Weyl class (for $x \to +\infty$), respectively, see Miller & Ross (1993) [97]. For example, power functions $|x|^{-\delta}$ with $\delta > \alpha > 0$ and exponential functions e^{cx} with $c > 0$ are of Liouville class (for $x \to -\infty$). For these functions we obtain, see **Problem (B.1)**,

$$\begin{cases} I_+^\alpha\,|x|^{-\delta} = \dfrac{\Gamma(\delta-\alpha)}{\Gamma(\delta)}\,|x|^{-\delta+\alpha}\,, \\ D_+^\alpha\,|x|^{-\delta} = \dfrac{\Gamma(\delta+\alpha)}{\Gamma(\delta)}\,|x|^{-\delta-\alpha}\,, \end{cases} \quad \delta > \alpha > 0\,, \quad x < 0\,, \qquad (B.3)$$

and, see **Problem (B.2)**,

$$\begin{cases} I_+^\alpha\,\mathrm{e}^{cx} = c^{-\alpha}\,\mathrm{e}^{cx}\,, \\ D_+^\alpha\,\mathrm{e}^{cx} = c^{\alpha}\,\mathrm{e}^{cx}\,, \end{cases} \quad c > 0\,, \quad x \in \mathbb{R}. \qquad (B.4)$$

[2]In fact, Liouville considered in a series of papers from 1832 to 1837 the integrals of progressive type (B.1), see *e.g.* [73, 74, 75]. On the other hand, H. Weyl (1917) [130] arrived at the regressive integrals of type (B.2) indirectly by defining fractional integrals suitable for periodic functions.

Also for the Liouville and Weyl fractional integrals we can state the corresponding *semigroup property*, see **Problem (B.3)**,

$$I_+^\alpha\, I_+^\beta = I_+^{\alpha+\beta}\,, \quad I_-^\alpha\, I_-^\beta = I_-^{\alpha+\beta}\,, \quad \alpha\,,\,\beta \ge 0\,, \qquad (B.5)$$

where, for complementation, we have defined $I_+^0 = I_-^0 := \mathbb{I}$ (Identity operator). For more detals on Liouville-Weyl fractional integrals we refer to Miller (1975) [96], Samko, KIlbas & Marichev [117] and Miller & Ross [97].

For the definition of the *Liouville-Weyl fractional derivatives of order* α we follow the scheme adopted in the previous section for bounded intervals. Having introduced the positive integer m so that $m-1 < \alpha \le m$ we define

$$\begin{cases} D_+^\alpha\, \phi(x) := D^m\, I_+^{m-\alpha}\, \phi(x)\,, & -\infty < x < b\,, \\ D_-^\alpha\, \phi(x) := (-1)^m\, D^m\, I_-^{m-\alpha}\, \phi(x)\,, & a < x < +\infty\,, \end{cases} \quad (m-1 < \alpha \le m)\,, \quad (B.6)$$

with $D_+^0 = D_-^0 = \mathbb{I}$. In fact we easily recognize using (B.5)-(B-6) the fundamental property

$$D_+^\alpha\, I_+^\alpha = \mathbb{I} = (-1)^m D_-^\alpha\, I_-^\alpha\,. \qquad (B.7)$$

The explicit expressions for the "proper" LIouville and Weyl fractional derivatives $(m-1 < \alpha < m)$ read

$$D_+^\alpha\, \phi(x) = \frac{1}{\Gamma(m-\alpha)}\, \frac{d^m}{dx^m} \int_{-\infty}^x (x-\xi)^{m-\alpha-1}\, \phi(\xi)\, d\xi\,, \quad x \in \mathbb{R}, \qquad (B.8)$$

$$D_-^\alpha\, \phi(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\, \frac{d^m}{dx^m} \int_x^{+\infty} (\xi-x)^{m-\alpha-1}\, \phi(\xi)\, d\xi\,, \quad x \in \mathbb{R}. \qquad (B.9)$$

Because of the unbounded intervals of integration, fractional integrals and derivatives of Liouville and Weyl type can be (successfully) handled via Fourier transform and related theory of pseudo-differential operators, that, as we shall see, simplifies their treatment. For this purpose, let us now recall our notations and the relevant results concerning the Fourier transform.

Let

$$\widehat{\phi}(\kappa) = \mathcal{F}\,\{\phi(x);\kappa\} = \int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x}\, \phi(x)\, dx\,, \quad \kappa \in \mathbb{R}, \qquad (B.10)$$

be the Fourier transform of a sufficiently well-behaved function $\phi(x)$, and let

$$\phi(x) = \mathcal{F}^{-1}\left\{\widehat{\phi}(\kappa);x\right\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-i\kappa x}\, \widehat{\phi}(\kappa)\, d\kappa\,, \quad x \in \mathbb{R}, \qquad (B.11)$$

denotes the inverse Fourier transform[3].

In this framework we also consider the class of pseudo-differential operators of which the ordinary repeated integrals and derivatives are special cases. A pseudo-differential operator A, acting with respect to the variable $x \in \mathbb{R}$, is defined through its Fourier representation, namely

$$\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, A\,\phi(x)\,dx = \widehat{A}(\kappa)\,\widehat{\phi}(\kappa)\,, \qquad (B.12)$$

where $\widehat{A}(\kappa)$ is referred to as symbol of A. An often applicable practical rule is

$$\widehat{A}(\kappa) = \left(A\,\mathrm{e}^{-i\kappa x}\right)\,\mathrm{e}^{+i\kappa x}\,, \quad \kappa \in \mathbb{R}, \qquad (B.13)$$

see **Problem (B.4)**. If B is another pseudo-differential operator, then we have

$$\widehat{A\,B}(\kappa) = \widehat{A}(\kappa)\,\widehat{B}(\kappa)\,. \qquad (B.14)$$

For the sake of convenience we adopt the notation $\stackrel{\mathcal{F}}{\leftrightarrow}$ to denote the juxtaposition of a function with its Fourier transform and that of a pseudo-differential operator with its symbol, namely

$$\phi(x) \stackrel{\mathcal{F}}{\leftrightarrow} \widehat{\phi}(\kappa)\,, \quad A \stackrel{\mathcal{F}}{\leftrightarrow} \widehat{A}\,. \qquad (B.15)$$

We now consider the pseudo-differential operators represented by the Liouville-Weyl fractional integrals and derivatives, Of course we assume that the integrals entering their definitions are in a proper sense, in order to ensure that the resulting functions of x can be Fourier transformable in ordinary or generalized sense.

The symbols of the fractional Liouville-Weyl integrals and derivatives can be easily derived according to, see **Problems (B.5), (B6)**,

$$\begin{cases} \widehat{I_{\pm}^{\alpha}} = (\mp i\kappa)^{-\alpha} = |\kappa|^{-\alpha}\,\mathrm{e}^{\pm i\,(\mathrm{sign}\,\kappa)\,\alpha\pi/2}\,, \\ \widehat{D_{\pm}^{\alpha}} = (\mp i\kappa)^{+\alpha} = |\kappa|^{+\alpha}\,\mathrm{e}^{\mp i\,(\mathrm{sign}\,\kappa)\,\alpha\pi/2}\,. \end{cases} \qquad (B.16)$$

Based on a former idea by Marchaud, see *e.g.* Marchaud (1927) [91], Samko, Kilbas & Marichev (1993) [117], Hilfer (1997) [56], we now point out purely integral

[3] In the ordinary theory of the Fourier transform the integral in (B.10) is assumed to be a "Lebesgue integral" whereas the one in (B.11) can be the "principal value" of a "generalized integral". In fact, $\phi(x) \in L_1(\mathbb{R})$, necessary for writing (B.10), is not sufficient to ensure $\widehat{\phi}(\kappa) \in L_1(\mathbb{R})$. However, we allow for an extended use of the Fourier transform which includes Dirac-type generalized functions: then the above integrals must be properly interpreted in the framework of the theory of distributions.

expressions for $D_\pm^\alpha$ which are alternative to the integro-differential expressions (B.8) and (B.9).

We limite ourselves to the case $0<\alpha<1$. Let us first consider from Eq. (B.8) the progressive derivative

$$D_+^\alpha = \frac{d}{dx}\, I_+^{1-\alpha}\,, \quad 0<\alpha<1\,. \tag{$B.17$}$$

We have, see Hilfer (1997) [56],

$$\begin{aligned} D_+^\alpha\,\phi(x) &= \frac{d}{dx}\, I_+^{1-\alpha}\,\phi(x) \\ &= \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int_{-\infty}^{x} (x-\xi)^{-\alpha}\,\phi(\xi)\,d\xi \\ &= \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int_{0}^{\infty} \xi^{-\alpha}\,\phi(x-\xi)\,d\xi \\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left[\phi'(x-\xi)\int_\xi^\infty \frac{d\eta}{\eta^{1+\alpha}}\right] d\xi\,, \end{aligned}$$

so that, interchanging the order of integration, see **Problem (B.7)**,

$$D_+^\alpha\,\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{\phi(x)-\phi(x-\xi)}{\xi^{1+\alpha}}\,d\xi\,, \quad 0<\alpha<1\,. \tag{$B.18$}$$

Here ϕ' denotes the first derivative of ϕ with respect to its argument. The coefficient in front to the integral of (B.18) can be re-written, using known formulas for the Gamma function, as

$$\frac{\alpha}{\Gamma(1-\alpha)} = -\frac{1}{\Gamma(-\alpha)} = \Gamma(1+\alpha)\,\frac{\sin\,\alpha\pi}{\pi}\,. \tag{$B.19$}$$

Similarly we get for the regressive derivative

$$D_-^\alpha = -\frac{d}{dx}\, I_-^{1-\alpha}\,, \quad 0<\alpha<1\,, \tag{$B.20$}$$

$$D_-^\alpha\,\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{\phi(x)-\phi(x+\xi)}{\xi^{1+\alpha}}\,d\xi\,, \quad 0<\alpha<1\,. \tag{$B.21$}$$

Similar results can be given for $\alpha\in(m-1,m)\,,\ m\in\mathbb{N}$.

C. The Riesz-Feller Fractional Calculus

The purpose of this section is to combine the Liouville-Weyl fractional integrals and derivatives in order to obtain the pseudo-differential operators considered around the 1950's by Marcel Riesz [111] and William Feller [38]. In particular the Riesz-Feller fractional derivatives will be used later to generalize the standard diffusion equation by replacing the second-order space derivative. So doing wse shall generate all the (symmetric and non symmetric) Lévy stable probability densities according to our parametrization.

The Riesz fractional integrals and derivatives

The Liouville-Weyl fractional integrals can be combined to give rise to the *Riesz fractional integral* (usually called *Riesz potential*) of order α, defined as

$$I_0^\alpha\,\phi(x) = \frac{I_+^\alpha\phi(x) + I_-^\alpha\phi(x)}{2\,\cos(\alpha\pi/2)} = \frac{1}{2\,\Gamma(\alpha)\,\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty}|x-\xi|^{\alpha-1}\,\phi(\xi)\,d\xi\,, \qquad (C.1)$$

for any positive α with the exclusion of odd integer numbers for which $\cos(\alpha\pi/2)$ vanishes. The symbol of the Riesz potential turns out to be

$$\widehat{I_0^\alpha} = |\kappa|^{-\alpha}\,, \quad \kappa\in\mathbb{R}, \qquad \alpha>0\,, \quad \alpha\neq 1\,,3\,,5\,\dots\,. \qquad (C.2)$$

In fact, recalling the symbols of the Liouville-Weyl fractional integrals, see Eq. (B.16), we obtain

$$\widehat{I_+^\alpha} + \widehat{I_-^\alpha} = \left[\frac{1}{(-i\kappa)^\alpha} + \frac{1}{(+i\kappa)^\alpha}\right] = \frac{(+i)^\alpha + (-i)^\alpha}{|\kappa|^\alpha} = \frac{2\,\cos(\alpha\pi/2)}{|\kappa|^\alpha}\,.$$

We note that, at variance with the Liouville fractional integral, the Riesz potential has the semigroup property only in restricted ranges, *e.g.*

$$I_0^\alpha I_0^\beta = I_0^{\alpha+\beta} \quad \text{for} \quad 0<\alpha<1\,, \quad 0<\beta<1\,, \quad \alpha+\beta<1\,. \qquad (C.3)$$

From the Riesz potential we can define by analytic continuation the *Riesz fractional derivative* D_0^α, including also the singular case $\alpha = 1$, by formally setting $D_0^\alpha := -I_0^{-\alpha}$, namely, in terms of symbols,

$$\widehat{D_0^\alpha} := -|\kappa|^\alpha\,. \qquad (C.4)$$

We note that the minus sign has been put in order to recover for $\alpha = 2$ the standard second derivative. Indeed, noting that

$$-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}\,, \qquad (C.5)$$

we recognize that the Riesz fractional derivative of order α is the opposite of the $\alpha/2$-power of the positive definite operator $-\frac{d^2}{dx^2}$

$$D_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2} . \qquad (C.6)$$

We also note that the two Liouville fractional derivatives are related to the α-power of the first order differential operator $D = \frac{d}{dx}$. We note that it was Bochner (1949) [13] who first introduced the fractional powers of the Laplacian to generalize the diffusion equation.

Restricting our attention to the range $0 < \alpha \le 2$ the explicit expression for the Riesz fractional derivative turns out to be

$$D_0^\alpha\, \phi(x) = \begin{cases} -\dfrac{D_+^\alpha\, \phi(x) + D_-^\alpha\, \phi(x)}{2\, \cos(\alpha\pi/2)} & \text{if } \ \alpha \neq 1\,, \\ -D\, H\, \phi(x)\,, & \text{if } \ \alpha = 1\,, \end{cases} \qquad (C.7)$$

where H denotes the Hilbert transform operator defined by

$$H\, \phi(x) := \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{x-\xi}\, d\xi = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(x-\xi)}{\xi}\, d\xi\,, \qquad (C.8)$$

the integral understood in the Cauchy principal value sense. Incidentally, we note that $H^{-1} = -H$. By using the practical rule (B.13) we can derive the symbol of H , namely, see **Problem (C.1)**,

$$\widehat{H} \;=\; i\, \mathrm{sign}\, \kappa \quad \kappa \in \mathbb{R}. \qquad (C.9)$$

The expressions in (C.7) can be easily verified by manipulating with symbols of "good" operators as below

$$\widehat{D_0^\alpha} = -\widehat{I_0^{-\alpha}} \;=\; -|\kappa|^\alpha = \begin{cases} -\dfrac{(-i\kappa)^\alpha + (+i\kappa)^\alpha}{2\, \cos(\alpha\pi/2)} = -|\kappa|^\alpha\,, & \text{if } \ \alpha \neq 1\,, \\ +i\kappa \cdot i\mathrm{sign}\, \kappa = -\kappa\, \mathrm{sign}\, \kappa = -|\kappa|\,, & \text{if } \ \alpha = 1\,. \end{cases}$$

In particular, from (C.7) we recognize that

$$D_0^2 = \frac{1}{2}\left(D_+^2 + D_-^2\right) = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{d^2}{dx^2}\right) = \frac{d^2}{dx^2}\,, \quad \text{but} \quad D_0^1 \neq \frac{d}{dx}\,.$$

We also recognize that the symbol of D_0^α $(0 < \alpha \le 2)$ is just the *cumulative function* (logarithm of the characteristic function) of a *symmetric* Lévy stable *pdf*, see *e.g.* Feller (1971) [39], Sato (1999) [119].

We would like to mention the "illuminating" notation introduced by Zaslavsky, see *e.g.* Saichev & Zaslavsky (1997) [116] to denote our Liouville and Riesz fractional derivatives

$$D_{\pm}^{\alpha} = \frac{d^{\alpha}}{d(\pm x)^{\alpha}}\,, \quad D_0^{\alpha} = \frac{d^{\alpha}}{d|x|^{\alpha}}\,, \quad 0 < \alpha \le 2\,. \qquad (C.10)$$

Recalling from (C.7) the fractional derivative in Riesz's sense

$$D_0^{\alpha}\,\phi(x) := -\frac{D_+^{\alpha}\,\phi(x) + D_-^{\alpha}\,\phi(x)}{2\,\cos(\alpha\pi/2)}\,, \quad 0 < \alpha < 1\,, \quad 1 < \alpha < 2\,,$$

and using (B.18) and (B.21) we get for it the following regularized representation, valid also in $\alpha = 1$,

$$D_0^{\alpha}\,\phi(x) = \Gamma(1+\alpha)\,\frac{\sin(\alpha\pi/2)}{\pi}\int_0^{\infty}\frac{\phi(x+\xi) - 2\phi(x) + \phi(x-\xi)}{\xi^{1+\alpha}}\,d\xi\,, \qquad (C.11)$$

$$0 < \alpha < 2\,.$$

We note that Eq. (C.11) has recently been derived by Gorenflo & Mainardi, see [51] and improves the corresponding formula in the book by Samko, Kilbas & Marichev (1993) [117] which is not valid for $\alpha = 1$.

The Feller fractional integrals and derivatives

A generalization of the Riesz fractional integral and derivative has been proposed by Feller (1952) [38] in a pioneering paper, recalled by Samko, Kilbas & Marichev (1993) [117], but only recently revised and used by Gorenflo & Mainardi (1998) [49]. Feller's intention was indeed to generalize the second order space derivative entering the standard diffusion equation by a pseudo-differential operator whose symbol is the *cumulative function* (logarithm of the characteristic function) of a general Lévy stable *pdf* according to his parameterization.

Let us now show how to obtain the Feller derivative by inversion of a properly generalized Riesz potential, later called *Feller potential* by Samko, Kilbas & Marichev (1993) [117]. Using our notation we define the Feller potential I_{θ}^{α} by its symbol obtained from the Riesz potential by a suitable "rotation" by an angle $\theta\,\pi/2$ properly restricted, *i.e.*

$$\widehat{I_{\theta}^{\alpha}}(\kappa) = |\kappa|^{-\alpha}\,\mathrm{e}^{-i\,(\mathrm{sign}\;\kappa)\,\theta\pi/2}\,, \quad |\theta| \le \begin{cases} \alpha & \text{if } 0 < \alpha < 1\,, \\ 2-\alpha & \text{if } 1 < \alpha \le 2\,, \end{cases}\,, \qquad (C.12)$$

with $\kappa\,,\,\theta \in \mathbb{R}$. Like for the Riesz potential the case $\alpha = 1$ is here omitted. The integral representation of I_{θ}^{α} turns out to be, see **Problem (C.2)**,

$$I_{\theta}^{\alpha}\,\phi(x) = c_-(\alpha,\theta)\,I_+^{\alpha}\,\phi(x) + c_+(\alpha,\theta)\,I_-^{\alpha}\,\phi(x)\,, \qquad (C.13)$$

where, if $0 < \alpha < 2\,,\ \alpha \neq 1\,,$

$$c_+(\alpha,\theta) = \frac{\sin\,[(\alpha-\theta)\,\pi/2]}{\sin\,(\alpha\pi)}\,, \qquad c_-(\alpha,\theta) = \frac{\sin\,[(\alpha+\theta)\,\pi/2]}{\sin(\alpha\pi)}\,, \tag{C.14}$$

and, by passing to the limit (with $\theta = 0$)

$$c_+(2,0) = c_-(2,0) = -1/2\,. \tag{C.15}$$

In the particular case $\theta = 0$ we get

$$c_+(\alpha,0) = c_-(\alpha,0) = \frac{1}{2\,\cos\,(\alpha\pi/2)}\,, \tag{C.16}$$

and thus, from (C.13) and (C.16) we recover the Riesz potential (C.1). Like the Riesz potential also the Feller potential has the (range-restricted) semigroup property, *e.g.*

$$I_\theta^\alpha I_\theta^\beta = I_\theta^{\alpha+\beta} \quad \text{for} \quad 0 < \alpha < 1\,, \quad 0 < \beta < 1\,, \quad \alpha+\beta < 1\,. \tag{C.17}$$

From the Feller potential we can define by analytical continuation the *Feller fractional derivative* D_θ^α, including also the singular case $\alpha = 1$, by setting

$$D_\theta^\alpha := -I_\theta^{-\alpha}\,,$$

so

$$\widehat{D_\theta^\alpha}(\kappa) = -|\kappa|^\alpha\,\mathrm{e}^{+i\,(\mathrm{sign}\ \kappa)\,\theta\pi/2}\,, \quad |\theta| \le \begin{cases} \alpha & \text{if } \ 0 < \alpha \le 1\,, \\ 2-\alpha & \text{if } \ 1 < \alpha \le 2\,. \end{cases} \tag{C.18}$$

Since for D_θ^α the case $\alpha = 1$ is included, the condition for θ in (C.18) can be shortened into

$$|\theta| \le \min\,\{\alpha, 2-\alpha\}\,, \quad 0 < \alpha \le 2\,.$$

We note that the allowed region for the parameters α and θ turns out to be a diamond in the plane $\{\alpha\,,\,\theta\}$ with vertices in the points $(0,0)$, $(1,1)$, $(2,0)$, $(1,-1)$, see Fig. 1. We call it the *Feller-Takayasu diamond*, since we like to also honour Takayasu, who in his 1990 book [123] first gave the diamond representation for the Lévy stable distributions.

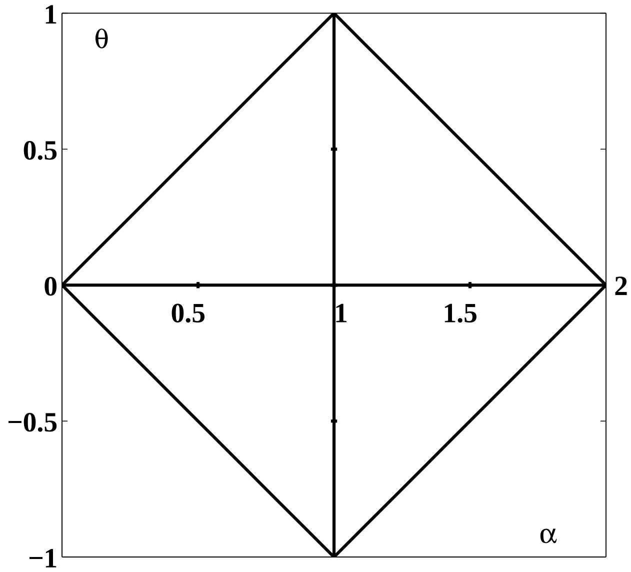

Fig. 1 – The Feller-Takayasu diamond

The representation of $D_\theta^\alpha\,\phi(x)$ can be obtained from the previous considerations. We have

$$D_\theta^\alpha\,\phi(x) = \begin{cases} -\,\left[c_+(\alpha,\theta)\,D_+^\alpha + c_-(\alpha,\theta)\,D_-^\alpha\right]\,\phi(x)\,, & \text{if } \alpha \neq 1\,, \\ \\ \left[\cos(\theta\pi/2)\,D_0^1 + \sin(\theta\pi/2)\,D\right]\,\phi(x)\,, & \text{if } \alpha = 1\,. \end{cases} \qquad (C.19)$$

For $\alpha \neq 1$ it is sufficient to note that $c_\mp(-\alpha,\theta) = c_\pm(\alpha,\theta)$. For $\alpha = 1$ we need to recall the symbols of the operators D and $D_0^1 = -DH$, namely $\hat{D} = (-i\kappa)$ and $\widehat{D_0^1} = -|\kappa|$, and note that

$$\begin{aligned} \widehat{D_\theta^1} &= -|\kappa|\,\mathrm{e}^{+i\,(\text{sign }\kappa)\,\theta\pi/2} = -\,|\kappa|\,\cos(\theta\pi/2) - (i\kappa)\,\sin(\theta\pi/2) \\ &= \cos(\theta\pi/2)\,\widehat{D_0^1} + \sin(\theta\pi/2)\,\widehat{D}\,. \end{aligned}$$

We note that in the extremal cases of $\alpha = 1$ we get

$$D_{\pm 1}^1 = \pm D = \pm\frac{d}{dx}\,. \qquad (C.20)$$

We also note that the representation by hyper-singular integrals for $0 < \alpha < 2$ (now excluding the cases $\{\alpha = 1\,,\ \theta \neq 0\}$) can be obtained by using (B.18) and (B.21) in the first equation of (C.19).

We get

$$D_\theta^\alpha\,\phi(x) = \frac{\Gamma(1+\alpha)}{\pi}\,\left\{\sin\left[(\alpha+\theta)\pi/2\right]\int_0^\infty \frac{\phi(x+\xi)-\phi(x)}{\xi^{1+\alpha}}\,d\xi \right.$$
$$\left. +\,\sin\left[(\alpha-\theta)\pi/2\right]\int_0^\infty \frac{\phi(x-\xi)-\phi(x)}{\xi^{1+\alpha}}\,d\xi\right\}\,, \qquad (C.21)$$

which reduces to (C.11) for $\theta = 0$.

For later use we find it convenient to return to the "weight" coefficients $c_\pm(\alpha,\theta)$ in order to outline some properties along with some particular expressions, which can be easily obtained from (C.14) with the restrictions on θ given in (C.12). We obtain

$$c_\pm \begin{cases} \ge 0\,, & \text{if } 0<\alpha<1\,,\\ \le 0\,, & \text{if } 1<\alpha\le 2\,,\end{cases} \qquad (C.22)$$

and

$$c_+ + c_- = \frac{\cos(\theta\pi/2)}{\cos(\alpha\pi/2)}\begin{cases} >0\,, & \text{if } 0<\alpha<1\,,\\ <0\,, & \text{if } 1<\alpha\le 2\,.\end{cases} \qquad (C.23)$$

In the *extremal* cases we find

$$0<\alpha<1\,,\ \begin{cases} c_+=1\,,\ c_-=0\,, & \text{if } \theta=-\alpha\,,\\ c_+=0\,,\ c_-=1\,, & \text{if } \theta=+\alpha\,,\end{cases} \qquad (C.24)$$

$$1<\alpha<2\,,\ \begin{cases} c_+=0\,,\ c_-=-1\,, & \text{if } \theta=-(2-\alpha)\,,\\ c_+=-1\,,\ c_-=0\,, & \text{if } \theta=+(2-\alpha)\,.\end{cases} \qquad (C.25)$$

In view of the relation of the Feller operators in the framework of *stable* probability density functions, we agree to refer to θ as to the *skewness* parameter.

We must note that in his original paper Feller (1952) [38] used a skewness parameter δ different from our θ; the potential introduced by Feller is such that

$$\widehat{I_\delta^\alpha} = \left(|\kappa|\,\mathrm{e}^{\,-i\,(\mathrm{sign}\,\kappa)\,\delta}\right)^{-\alpha}\,,\quad \delta = -\frac{\pi}{2}\,\frac{\theta}{\alpha}\,,\quad \theta = -\frac{2}{\pi}\,\alpha\delta\,. \qquad (C.26)$$

In their recent book, Uchaikin & Zolotarev (1999) [125] have adopted Feller's convention, but using the letter θ for Feller's δ.

D. The Abel-Riemann Fractional Calculus

In this Section we consider sufficiently well-behaved functions $\psi(t)$ $(t \in \mathbb{R}_0^+)$ with Laplace transform defined as

$$\widetilde{\psi}(s) = \mathcal{L}\{\psi(t); s\} = \int_0^\infty e^{-st}\,\psi(t)\,dt\,, \quad \Re(s) > a_\psi\,, \tag{D.1}$$

where a_ψ denotes the abscissa of convergence. The inverse Laplace transform is then given as

$$\psi(t) = \mathcal{L}^{-1}\left\{\widetilde{\psi}(s); t\right\} = \frac{1}{2\pi i}\int_{Br} e^{st}\,\widetilde{\psi}(s)\,ds\,, \quad t > 0\,, \tag{D.2}$$

where Br is a Bromwich path[4], namely $\{\gamma - i\infty, \gamma + i\infty\}$ with $\gamma > a_\psi$.

It may be convenient to consider $\psi(t)$ as "causal" function in $\mathbb{R}$ namely vanishing for all $t < 0$. For the sake of convenience we adopt the notation $\stackrel{\mathcal{L}}{\leftrightarrow}$ to denote the juxtaposition of a function with its Laplace transform, with its symbol, namely

$$\psi(t) \stackrel{\mathcal{L}}{\leftrightarrow} \widetilde{\psi}(s)\,. \tag{D.3}$$

The Abel-Riemann fractional integrals and derivatives

We first define the Abel-Riemann (A-R) fractional integral and derivative of any order $\alpha > 0$ for a generic (well-behaved) function $\psi(t)$ with $t \in \mathbb{R}^+$.

For the A-R *fractional integral* (of order α) we have

$$J^\alpha\,\psi(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\psi(\tau)\,d\tau\,, \quad t > 0 \quad \alpha > 0\,. \tag{D.4}$$

For complementation we put $J^0 := \mathbb{I}$ (Identity operator), as it can be justified by passing to the limit $\alpha \to 0$.

The A-R integrals possess the semigroup property, see **Problem (D.1)**,

$$J^\alpha\,J^\beta = J^{\alpha+\beta}\,, \text{ for all } \alpha\,,\, \beta \geq 0\,. \tag{D.5}$$

The A-R *fractional derivative* (of order $\alpha > 0$) is defined as the left-inverse operator of the corresponding A-R fractional integral (of order $\alpha > 0$), *i.e.*

$$D^\alpha\,J^\alpha = \mathbb{I}\,. \tag{D.6}$$

[4] In the ordinary theory of the Laplace transform the condition $\psi(t) \in L_{loc}(\mathbb{R}^+)$, is necessarily required, and the Bromwich integral is intended in the "principal value" sense. However, we allow an extended use of the theory of Laplace transform which includes Dirac-type generalized functions: then the above integrals must be properly interpreted in the framework of the theory of distributions.

Therefore, introducing the positive integer m such that $m-1 < \alpha \le m$ and noting that $(D^m\, J^{m-\alpha})\, J^\alpha = D^m\, (J^{m-\alpha}\, J^\alpha) = D^m\, J^m = \mathbb{I}$, we define

$$D^\alpha := D^m\, J^{m-\alpha}\,, \quad m-1 < \alpha \le m\,, \tag{$D.7$}$$

namely

$$D^\alpha \psi(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\, \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{\psi(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\,, & m-1 < \alpha < m\,, \\ \dfrac{d^m}{dt^m}\, \psi(t)\,, & \alpha = m\,. \end{cases} \tag{$D.8$}$$

For complementation we put $D^0 := \mathbb{I}$. For $\alpha \to m^-$ we thus recover the standard derivative of order m but the integral formula loses its meaning for $\alpha = m$.

By using the properties of the Eulerian beta and gamma functions it is easy to show the effect of our operators J^α and D^α on the power functions: we have, see **Problem (D.2)**,

$$\begin{cases} J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \\ D^\alpha\, t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,, \end{cases} \quad t > 0\,, \quad \alpha \ge 0\,, \quad \gamma > -1\,. \tag{$D.9$}$$

These properties are of course a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha\, \psi(t)$ is not zero for the constant function $\psi(t) \equiv 1$ if $\alpha \notin \mathbb{N}$. In fact, the second formula in (D.9) with $\gamma = 0$ teaches us that

$$D^\alpha\, 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad \alpha \ge 0\,, \quad t > 0\,. \tag{$D.10$}$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbb{N}$, due to the poles of the gamma function in the points $0, -1, -2, \ldots$.

The Caputo fractional derivative

We now observe that an alternative definition of the fractional derivative has been introduced in the late sixties by Caputo (1967), (1969) [17, 18] and soon later adopted in physics to deal long-memory visco-elastic processes by Caputo and Mainardi see [26, 27], and, for a more recent review, Mainardi (1997) [84].

In this case the fractional derivative, denoted by D_*^α, is defined by exchanging the operators $J^{m-\alpha}$ and D^m in the classical definition (D.7), namely

$$D_*^\alpha := J^{m-\alpha}\, D^m\,, \quad m-1 < \alpha \le m\,. \tag{$D.11$}$$

In the literature, after the appearance in 1999 of the book by Podlubny [107], such derivative is known simply as *Caputo derivative.* Based on (D.11) we have

$$D_*^\alpha\,\psi(t) \,:=\, \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \frac{\psi^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau\,, & m-1<\alpha<m\,,\\ \dfrac{d^m}{dt^m}\psi(t)\,, & \alpha=m\,. \end{cases} \qquad (D.12)$$

For $m-1<\alpha<m$ the definition (D.11) is of course more restrictive than (D.7), in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha\,\psi(t) = D^m\,J^{m-\alpha}\,\psi(t) \neq J^{m-\alpha}\,D^m\,\psi(t) = D_*^\alpha\,\psi(t)\,, \qquad (D.13)$$

unless the function $\psi(t)$ along with its first $m-1$ derivatives vanishes at $t=0^+$. In fact, assuming that the passage of the m-th derivative under the integral is legitimate, one recognizes that, for $m-1<\alpha<m$ and $t>0$, see **Problem (D.3)**,

$$D^\alpha\,\psi(t) = D_*^\alpha\,\psi(t) + \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\,\psi^{(k)}(0^+)\,. \qquad (D.14)$$

As noted by Samko, Kilbas & Marichev [117] and Butzer & Westphal [16] the identity (D.14) was considered by Liouville himself (but not used for an alternative definition of fractional derivative).

Recalling the fractional derivative of the power functions, see the second equation in (D.9), we can rewrite (D.14) in the equivalent form

$$D^\alpha\left(\psi(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}\,\psi^{(k)}(0^+)\right) = D_*^\alpha\,\psi(t)\,. \qquad (D.15)$$

The subtraction of the Taylor polynomial of degree $m-1$ at $t=0^+$ from $\psi(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero can be easily recognized,

$$D_*^\alpha 1 \equiv 0\,,\quad \alpha>0\,. \qquad (D.16)$$

As a consequence of (D.15) we can interpret the Caputo derivative as a sort of regularization of the R-L derivative as soon as the values $\psi^k(0^+)$ are finite; in this sense such fractional derivative was independently introduced in 1968 by Dzherbashyan and Nersesian [34], as pointed out in interesting papers by Kochubei (1989), (1990), see [69, 70]. In this respect the regularized fractional derivative is sometimes referred to as the *Caputo-Dzherbashyan derivative.*

We now explore the most relevant differences between the two fractional derivatives (D.7) and (D.11). We agree to denote (D.11) as the *Caputo fractional derivative* to distinguish it from the standard A-R fractional derivative (D.7). We observe, again by looking at se cond equation in (D.9), that $D^\alpha t^{\alpha-k} \equiv 0\,,\; t>0$ for $\alpha>0$, and $k=1,2,\dots,m$. We thus recognize the following statements about functions which for $t>0$ admit the same fractional derivative of order α, with $m-1<\alpha\le m$, $m\in\mathbb{N}$,

$$D^\alpha\,\psi(t) = D^\alpha\,\phi(t) \iff \psi(t) = \phi(t) + \sum_{j=1}^{m} c_j\,t^{\alpha-j}\,, \qquad (D.17)$$

$$D_*^\alpha\,\psi(t) = D_*^\alpha\,\phi(t) \iff \psi(t) = \phi(t) + \sum_{j=1}^{m} c_j\,t^{m-j}\,, \qquad (D.18)$$

where the coefficients c_j are arbitrary constants.

For the two definitions we also note a difference with respect to the *formal* limit as $\alpha\to(m-1)^+$; from (D.7) and (D.11) we obtain respectively,

$$D^\alpha\,\psi(t) \to D^m\,J\,\psi(t) = D^{m-1}\,\psi(t)\,; \qquad (D.19)$$

$$D_*^\alpha\,\psi(t) \to J\,D^m\,\psi(t) = D^{m-1}\,\psi(t) - \psi^{(m-1)}(0^+)\,. \qquad (D.20)$$

We now consider the *Laplace transform* of the two fractional derivatives. For the A-R fractional derivative D^α the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k=1,2,\dots,m-1$. The corresponding rule reads, in our notation, see **Problem (D.4)**,

$$D^\alpha\,\psi(t) \overset{\mathcal{L}}{\leftrightarrow} s^\alpha\,\widetilde{\psi}(s) - \sum_{k=0}^{m-1} D^k\,J^{(m-\alpha)}\,\psi(0^+)\,s^{m-1-k}\,, \quad m-1<\alpha\le m\,. \qquad (D.21)$$

For the *Caputo* fractional derivative the Laplace transform technique requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k=1,2,\dots,m-1$, in analogy with the case when $\alpha=m$. In fact, noting that $J^\alpha\,D_*^\alpha = J^\alpha\,J^{m-\alpha}\,D^m = J^m\,D^m$, we have, see **Problem (D.5)**,

$$J^\alpha\,D_*^\alpha\,\psi(t) = \psi(t) - \sum_{k=0}^{m-1}\psi^{(k)}(0^+)\,\frac{t^k}{k!}\,, \qquad (D.22)$$

so we easily prove the following rule for the Laplace transform,

$$D_*^\alpha\,\psi(t) \overset{\mathcal{L}}{\leftrightarrow} s^\alpha\,\widetilde{\psi}(s) - \sum_{k=0}^{m-1}\psi^{(k)}(0^+)\,s^{\alpha-1-k}\,, \quad m-1<\alpha\le m\,. \qquad (D.23)$$

Indeed the result (D.23), first stated by Caputo (1969) [18], appears as the "natural" generalization of the corresponding well known result for $\alpha = m$.

Gorenflo & Mainardi (1997) [48] have pointed out the major utility of the Caputo fractional derivative in the treatment of differential equations of fractional order for *physical applications*. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, despite the fact that the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.

We note that the Caputo fractional derivative is not mentioned in the standard books of fractional calculus (including the encyclopaedic treatise by Samko, Kilbas & Marichev (1993)[117]) with the exception of the recent book by Podlubny (1999) [107], where this derivative is largely treated in the theory and applications. Several applications have also been treated by Caputo himself from the seventies up to nowadays, see *e.g.* [19, 20, 21, 22, 23, 24, 25].

E. The Grünwald-Letnikov Fractional Calculus

Grünwald [54] and Letnikov [71, 72] generallized the classical definition of derivatives of integer order as limits of difference quotients to the case of fractional derivatives. Let us give an outline of their approach in a way just appropriate for later use in the construction of discrete random walk models for fractional diffusion processes. For more detailed presentations, proofs of convergence, suitable function spaces etc, we refer the reader to Butzer & Westphal [16], Podlubny [107], Miller & Ross [97], Samko, Kilbas & Marichev [117]. It is noteworthy that also Oldham & Spanier in their pioneering book [105] made extensive use of this approach.

We need one essential discrete operator, namely the *shift operator* E^h which for $h \in \mathbb{R}$ in its application to a function $\phi(x)$ defined in $\mathbb{R}$, has the effect

$$E^h\,\phi(x) = \phi(x+h)\,. \tag{E.1}$$

Obviously the operators E^h, for $h \in \mathbb{R}$, possess the group property

$$E^{h_1}\,E^{h_2} = E^{h_2}\,E^{h_1} = E^{h_1+h_2}\,, \quad h_1, h_2 \in \mathbb{R}, \tag{E.2}$$

and it is for this property that that we put h in the place of an exponent rather than in the place of an index. In fact, we can formally write

$$E^h = \mathrm{e}^{hD} = \sum_{n=0}^{\infty} \frac{1}{n!}\,h^n\,D^n\,,$$

as a shorthand notation for the Taylor expansion of a function analytic at the point x.

In the sequel, if not explicitly said otherwise, we will assume

$$h > 0\,. \qquad (E.3)$$

We are now going to describe in some detail how to approximate our fractional derivatives, introduced in the previous sessions, like D^α_{a+}, $D^\alpha_{+}, \dots$ for $\alpha > 0$. The corresponding formulas with the minus sign in the index can then be obtained by consideration of symmetry.

It is convenient to introduce the *backward* difference operator Δ_h as

$$\Delta_h \,=\, \mathbb{I} - E^{-h}\,, \qquad (E.4)$$

whose process can, via the binomial theorem, readily be expressed in terms of the operators $E^{-kh} = \left(E^{-h}\right)^k$:

$$\Delta_h^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{-kh}\,, \quad n \in \mathbb{N}\,. \qquad (E.5)$$

For complemantation we also define $\Delta_h^0 = \mathbb{I}$.

It is well known that for sufficiently smooth functions for positive integer n

$$h^{-n}\,\Delta_h^n\,\phi(x+ch) = D^n\,\phi(x) + O(h)\,, \quad h \to 0^+\,. \qquad (E.6)$$

Here c stands for an arbitrary *fixed* real number (independent of h). The choice $c = 0$ or $c = n$ leads to completely one-sided (backward or forward, respectively) approximation, whereas the choice $c = n/2$ leads to approximation of second order accuracy, *i.e.* $O(h^2)$ in place of $O(h)$.

The Grünwald-Letnikov approximation in the Riemann-Liouville fractional calculs

The Grünwald-Letnikov approach to fractional differentiation consists in replacing in (E.5) and (E.6) the positive integer n by an arbitrary positive number α. Then, instead of the binomial formula, we have to use the binomial series, and we get (formally)

$$\Delta_h^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} E^{-kh}\,, \quad \alpha > 0\,. \qquad (E.7)$$

with the generalized binomial coefficients

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha-n+1)}\,.$$

If now $\phi(x)$ is a sufficiently well-behaved function (defined in $\mathbb{R}$) we have the limit relation

$$h^{-\alpha}\,\Delta_h^\alpha\,\phi(x+ch) \to D_+^\alpha\,\phi(x)\,,\quad h\to 0^+\,,\quad \alpha>0\,. \tag{E.8}$$

Here usually we take $c=0$, but in general a *fixed* real number c is allowed if ϕ is smooth enough (compare Vu Kim Tuan & Gorenflo [126]).

For sufficient conditions of convergence and also for convergence $h^{-\alpha}\,\Delta_h^\alpha\,\phi \to D_+^\alpha\,\phi$, in norms of appropriate Banach spaces instead of poinwise convergence the readers may consult the above-mentioned references. From [126] one can take that in the case $c=0$ convergence is pointwise ((as written in (E.8)) and of order $O(h)$ if $\phi \in C^{[\alpha]+2}(\mathbb{R})$ and all derivatives of ϕ up to the order $[\alpha]+3$ belong to $L_1(\mathbb{R})$,

In our later theory of random walk models for space-fractional diffusion such precise conditions are irrelevant because *ex posteriori* we can prove their convergence.

The Grünwald-Letnikov approximations will there serve us as a heuristic guide to constructional models. In fact, we will need these approximations just for $0<\alpha\le 2$, for $0<\alpha<1$ with $c=0$, for $1<\alpha\le 2$ with $c=1$, in (E.8) (The case $\alpha=1$ being singular in these special random walk models).

Before proceeding further, let us write in longhand notation (for the special choice $c=0$) the formula (E.8) in the form

$$h^{-\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\phi(x-kh) \to D_+^\alpha\,\phi(x)\,,\quad h\to 0^+\,,\quad \alpha>0\,, \tag{E.9}$$

and the corresponding formula

$$h^{-\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\phi(x+kh) \to D_-^\alpha\,\phi(x)\,,\quad h\to 0^+\,,\quad \alpha>0\,, \tag{E.10}$$

If $\alpha=n\in\mathbb{N}$, then $\binom{\alpha}{k}=0$ for $k>\alpha=n$, and we have finite sums instead of infinite series.

The Grünwald-Letnikov approach to the Riemann-Liouville derivative $D_{a+}^\alpha\,\phi(x)$ where $x>a>-\infty$ consistes in extending $\phi(x)$ by $\phi(x)=0$ for $x<a$ and then setting

$${}_{GL}D_{a+}^\alpha\,\phi(x) = \lim_{h\to 0^+} h^{-\alpha}\,\Delta_h^\alpha\,\phi(x)\,,\quad h\to 0^+\,,\quad x>a\,. \tag{E.11}$$

If $\phi(x)$ is smooth enough (in particular at $x=a$ where we require $\phi(a)=0$) we have

$${}_{GL}D_{a+}^\alpha\,\phi(x) = D_{a+}^\alpha\,\phi(x)\,, \tag{E.12}$$

either pointwhise or in a suitable function space.

Taking $x-a$ as a multiple of h, we can write the Grünwald-Letnikov as a finite sum

$$D_{a+}^\alpha\,\phi(x) \simeq h^{-\alpha}\,\Delta_h^\alpha\,\phi(x) = h^{-\alpha}\sum_{k=0}^{(x-a)/h}(-1)^k\binom{\alpha}{k}\phi(x-kh)\,. \tag{E.13}$$

One can show [see Samko, Kilbas & Marichev [117] Eq. (20.45) and compare our formula (D.9)]

$$ {}_{GL}D_{0+}^{\alpha}\, x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, x^{\gamma-\alpha} = D_{0+}^{\alpha}\, x^{\gamma} = D^{\alpha}\, x^{\gamma}\,, $$

with $x > 0$ and $\gamma > -1\,,\ \alpha > 0\,.$

We refrain for writing down the analogous formulas for approximation of the regressive derivative $D_{b-}^{\alpha}\,\phi(x)\,.$

Let us finally point out that by formally replacing α by $-\alpha$ we obtain approximation of the operators of fractional integration as I_{a+}^{α} with $\alpha > 0$, namely

$$ I_{0+}^{\alpha}\,\phi(x) \;\simeq\; h^{\alpha} \sum_{k=0}^{(x-a)/h} (-1)^k \binom{-\alpha}{k}\,\phi(x-kh)\,, \qquad (E.14) $$

where

$$ (-1)^k \binom{-\alpha}{k}\,\phi(x-kh) = \frac{(\alpha)_k}{k!} > 0\,. \qquad (E.15) $$

Here we have used the convention

$$ \frac{(\alpha)_k}{k!} = \prod_{j=0}^{k-1} = \alpha(\alpha-1)\ldots(\alpha-k+1)\,. $$

Remark concerning notation. In numerical analysis it is customary to write ∇_h (instead of Δ_h) for the backward difference operator $I - E^{-h}$ and to use Δ_h for the forward difference operator $E^h - I$, as in Gorenflo (1997) [44]. In our present work, however, we follow the notation employed by most workers in fractional calculus.

The Grünwald-Letnikov approximation in the Riesz-Feller fractional calculus

The Grünwald-Letnikov approximation for the fractional Liouville-Weyl derivatives D_{+}^{α}, D_{-}^{α} gives us an approximation to the Riesz-Feller derivatives D_{0}^{α}, D_{θ}^{α}, with $0 < \alpha < 2$, according to Eqs (C.7) and (C.19) (respectively), provided we disregard the singular case $\alpha = 1$,

F. The Mittag-Leffler functions

The Mittag-Leffler function is so named after the great Swedish mathematician Gosta Mittag-Leffler, who introduced and investigated it at the beginning of the 20-th century in a sequence of five notes [98, 99, 100, 101, 102]. In this Section we shall consider the Mittag-Leffler function and some of the related functions which are relevant for fractional evolution processes. It is our intention to provide a short reference-historical background and a review of the main properties of these functions, based on our papers, see [47], [85], [48], [84], [86].

Reference-historical background

We note that the Mittag-Leffler type functions, being ignored in the common books on special functions, are unknown to the majority of scientists. Even in the 1991 Mathematics Subject Classification these functions cannot be found. However the have now appeared in the new MSC scheme of the year 2000 under the number 33E12 ("Mittag-Leffler functions").

A description of the most important properties of these functions (with relevant references up to the fifties) can be found in the third volume of the Bateman Project [36], in the chapter $XVIII$ on *Miscellaneous Functions*. The treatises where great attention is devoted to them are those by Djrbashian (or Dzherbashian) (1966), (1993) [31, 32] We also recommend the classical treatise on complex functions by Sansone & Gerretsen (1960) [118]. The Mittag-Leffler functions are widely used in the books on fractional calculus and its applications, see *e.g.* Samko, Kilbas & Marichev (1987-93)[117], Podlubny (1999) [107], West, Bologna & Grigolini [127] Kilbas, Srivastava & Trujillo (2006)[67], Magin (2006) [78].

Since the times of Mittag-Leffler several scientists have recognized the importance of the functions named after him, providing interesting mathematical and physical applications which unfortunately are not much known. As pioneering works of mathematical nature in the field of fractional integral and differential equations, we like to quote those by Hille & Tamarkin (1930) [58] and Barret (1954) [10]. Hille & Tamarkin have provided the solution of the Abel integral equation of the second kind in terms of a Mittag-Leffler function, whereas Barret has expressed the general solution of the linear fractional differential equation with constant coefficients in terms of Mittag-Leffler functions. As former applications in physics we like to quote the contributions by Cole (1933) [29] in connection with nerve conduction, see also Davis (1936) [30],and by Gross (1947) [53] in connection with mechanical relaxation. Subsequently, Caputo & Mainardi (1971a), (1971b) [26, 27] have shown that Mittag-Leffler functions are present whenever derivatives of fractional order are introduced in the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of these functions for fractional viscoelastic models, see *e.g.* Mainardi (1997) [84].

The Mittag-Leffler functions $E_\alpha(z)$, $E_{\alpha,\beta}(z)$

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by its power series, which converges in the whole complex plane,

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \qquad (F.1)$$

It turns out that $E_\alpha(z)$ is an *entire function* of order $\rho = 1/\alpha$ and type 1, see **Problem (F.1)**. This property is still valid but with $\rho = 1/\Re(\alpha)$, if $\alpha \in \mathbb{C}$ with *positive real part*, as formerly noted by Mittag-Leffler himself in [101].

The Mittag-Leffler function provides a simple generalization of the exponential function to which it reduces for $\alpha = 1$. Other particular cases of (F.1), from which elementary functions are recovered, are

$$E_2\left(+z^2\right) = \cosh z, \qquad E_2\left(-z^2\right) = \cos z, \qquad z \in \mathbb{C}, \qquad (F.2)$$

and, see **Problem (F.2)**,

$$E_{1/2}(\pm z^{1/2}) = \mathrm{e}^z \left[1 + \mathrm{erf}\,(\pm z^{1/2})\right] = \mathrm{e}^z\, \mathrm{erfc}\,(\mp z^{1/2}), \quad z \in \mathbb{C}, \qquad (F.3)$$

where erf (erfc) denotes the (complementary) error function defined as

$$\mathrm{erf}\,(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2}\, du, \quad \mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z), \quad z \in \mathbb{C}. \qquad (F.4)$$

In (F.4) by $z^{1/2}$ we mean the principal value of the square root of z in the complex plane cut along the negative real semi-axis. With this choice $\pm z^{1/2}$ turns out to be positive/negative for $z \in \mathbb{R}^+$. A straightforward generalization of the Mittag-Leffler function, originally due to Agarwal in 1953 based on a note by Humbert, see [4], [61], [62], is obtained by replacing the additive constant 1 in the argument of the Gamma function in (F.1) by an arbitrary complex parameter β. For the generalized Mittag-Leffler function we agree to use the notation

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \qquad (F.5)$$

Particular simple cases are, see **Problem (F.3)**,

$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}, \quad E_{2,2}(z) = \frac{\sinh\,(z^{1/2})}{z^{1/2}}. \qquad (F.6)$$

We note that $E_{\alpha,\beta}(z)$ is still an entire function of order $\rho = 1/\alpha$ and type 1.

The Mittag-Leffler integral representation and asymptotic expansions

Many of the important properties of $E_\alpha(z)$ follow from Mittag-Leffler's *integral representation*

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\,\mathrm{e}^{\zeta}}{\zeta^\alpha - z}\,d\zeta\,, \quad \alpha > 0\,, \quad z \in \mathbb{C}\,, \tag{F.7}$$

where the path of integration Ha (the *Hankel path*) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\zeta| \le |z|^{1/\alpha}$ in the positive sense: $-\pi \le \arg\zeta \le \pi$ on Ha. To prove (F.7), expand the integrand in powers of ζ, integrate term-by-term, and use Hankel's integral for the reciprocal of the Gamma function, see **Problem (F.4)**.

The integrand in (F.7) has a branch-point at $\zeta = 0$. The complex ζ-plane is cut along the negative real semi-axis, and in the cut plane the integrand is single-valued: the principal branch of ζ^α is taken in the cut plane. The integrand has poles at the points $\zeta_m = z^{1/\alpha}\,\mathrm{e}^{2\pi\, i\, m/\alpha}$, m integer, but only those of the poles lie in the cut plane for which $-\alpha\,\pi < \arg z + 2\pi\, m < \alpha\,\pi$. Thus, the number of the poles inside Ha is either $[\alpha]$ or $[\alpha+1]$, according to the value of $\arg z$.

The most interesting properties of the Mittag-Leffler function are associated with its asymptotic developments as $z \to \infty$ in various sectors of the complex plane, see **Problem (F.5)**. These properties can be summarized as follows. For the case $0 < \alpha < 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha}\exp(z^{1/\alpha}) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \quad |z| \to \infty\,, \quad |\arg z| < \alpha\pi/2\,, \tag{F.8a}$$

$$E_\alpha(z) \sim -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \quad |z| \to \infty\,, \quad \alpha\pi/2 < \arg z < 2\pi - \alpha\pi/2\,. \tag{F.8b}$$

For the case $\alpha \ge 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha}\sum_m \exp\left(z^{1/\alpha}\mathrm{e}^{2\pi\, i\, m/\alpha}\right) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \quad |z| \to \infty\,, \tag{F.9}$$

where m takes all integer values such that $-\alpha\pi/2 < \arg z + 2\pi\, m < \alpha\pi/2$, and $\arg z$ can assume any value from $-\pi$ to $+\pi$.

From the asymptotic properties (F.8)-(F.9) and the definition of the order of an entire function, we infer that the Mittag-Leffler function is an *entire function of order* $1/\alpha$ for $\alpha > 0$; in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order. The Mittag-Leffler function also furnishes examples and counter-examples for the growth and other properties of entire functions of finite order, see **Problem (F.6)**.

Finally, the integral representation for the generalized Mittag-Leffler function reads

$$E_{\alpha.\beta}(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-\beta}\, \mathrm{e}^{\zeta}}{\zeta^{\alpha} - z}\, d\zeta\,, \quad \alpha > 0\,, \quad \beta \in \mathbb{C}\,, \quad z \in \mathbb{C}\,. \qquad (F.10)$$

The Laplace transform pairs related to the Mittag-Leffler functions

The Mittag-Leffler functions are connected to the Laplace integral through the equation, see **Problem (F.7)**.

$$\int_0^\infty \mathrm{e}^{-u}\, E_\alpha\,(u^\alpha\, z)\; du = \frac{1}{1-z}\,, \quad \alpha > 0\,. \qquad (F.11)$$

The integral at the L.H.S. was evaluated by Mittag-Leffler who showed that the region of its convergence contains the unit circle and is bounded by the line $\mathrm{Re}\, z^{1/\alpha} = 1$. Putting in (F.11) $u = st$ and $u^\alpha\, z = -a\, t^\alpha$ with $t \ge 0$ and $a \in \mathbb{C}$, and using the sign $\overset{\mathcal{L}}{\leftrightarrow}$ for the juxtaposition of a function depending on t with its Laplace transform depending on s, we get the following Laplace transform pairs

$$E_\alpha\,(-a\, t^\alpha) \overset{\mathcal{L}}{\leftrightarrow} \frac{s^{\alpha-1}}{s^\alpha + a}\,, \quad \Re\,(s) > |a|^{1/\alpha}\,. \qquad (F.12)$$

More generally one can show, see **Problem (F.8)**,

$$\int_0^\infty \mathrm{e}^{-u}\, u^{\beta-1}\, E_{\alpha,\beta}\,(u^\alpha\, z)\; du = \frac{1}{1-z}\,, \quad \alpha\,,\, \beta > 0\,, \qquad (F.13)$$

and

$$t^{\beta-1}\, E_{\alpha,\beta}\,(a\, t^\alpha) \overset{\mathcal{L}}{\leftrightarrow} \frac{s^{\alpha-\beta}}{s^\alpha - a}\,, \quad \Re\,(s) > |a|^{1/\alpha}\,. \qquad (F.14)$$

We note that the results (F.12) and (F.14) were used by Humbert (1953) [61] to obtain a number of functional relations satisfied by $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$.

Fractional relaxation and fractional oscillation

In our CISM Lecture notes, see Gorenflo & Mainardi (1997) [48], we have worked out the key role of the Mittag-Leffler type functions $E_\alpha\,(-a\, t^\alpha)$ in treating Abel integral equations of the second kind and fractional differential equations, so improving the former results by Hille & Tamarkin (1930) [58] and Barret (1954) [10], respectively. In particular, assuming $a > 0$, we have discussed the peculiar characters of these functions (*power-law decay*) for $0 < \alpha < 1$ and for $1 < \alpha < 2$ related to *fractional relaxation* and *fractional oscillation* processes, respectively, see also Mainardi (1996) [83] and Gorenflo & Mainardi [47].

Generally speaking, we consider the following differential equation of fractional order $\alpha > 0$,

$$D_*^\alpha\, u(t) = D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\, u^{(k)}(0^+) \right) = -u(t) + q(t)\,, \quad t > 0\,, \qquad (F.15)$$

where $u = u(t)$ is the field variable and $q(t)$ is a given function, continuous for $t \ge 0$. Here m is a positive integer uniquely defined by $m-1 < \alpha \le m$, which provides the number of the prescribed initial values $u^{(k)}(0^+) = c_k\,,\ k = 0, 1, 2, \ldots, m-1$. Implicit in the form of (F.15) is our desire to obtain solutions $u(t)$ for which the $u^{(k)}(t)$ are continuous for $t \ge 0$ for $k = 0, 1, 2, \ldots, m-1$. In particular, the cases of *fractional relaxation* and *fractional oscillation* are obtained for $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively

The application of the Laplace transform through the Caputo formula (D.19) yields

$$\widetilde{u}(s) = \sum_{k=0}^{m-1} c_k\, \frac{s^{\alpha-k-1}}{s^\alpha + 1} + \frac{1}{s^\alpha + 1}\, \widetilde{q}(s)\,. \qquad (F.16)$$

Then, using (F.12), we put for $k = 0, 1, \ldots, m-1$,

$$u_k(t) := J^k e_\alpha(t) \stackrel{\mathcal{L}}{\leftrightarrow} \frac{s^{\alpha-k-1}}{s^\alpha + 1}\,, \quad e_\alpha(t) := E_\alpha(-t^\alpha) \stackrel{\mathcal{L}}{\leftrightarrow} \frac{s^{\alpha-1}}{s^\alpha + 1}\,, \qquad (F.17)$$

and, from inversion of the Laplace transforms in (F.16), using $u_0(0^+) = 1$, we find

$$u(t) = \sum_{k=0}^{m-1} c_k\, u_k(t) - \int_0^t q(t-\tau)\, u_0'(\tau)\, d\tau\,. \qquad (F.18)$$

In particular, the formula (F.18) encompasses the solutions for $\alpha = 1\,, 2\,,$ since $e_1(t) = \exp(-t)\,,\ e_2(t) = \cos\, t$. When α is not integer, namely for $m-1 < \alpha < m$, we note that $m-1$ represents the integer part of α (usually denoted by $[\alpha]$) and m the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. Thus the m functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0, 1, \ldots, m-1$ represent those particular solutions of the *homogeneous* equation which satisfy the initial conditions $u_k^{(h)}(0^+) = \delta_{k\,h}\,,\ h, k = 0, 1, \ldots, m-1$, and therefore they represent the *fundamental solutions* of the fractional equation (F.15), in analogy with the case $\alpha = m$. Furthermore, the function $u_\delta(t) = -u_0'(t) = -e_\alpha'(t)$ represents the *impulse-response solution.*

In **Problem (F.9)** we derive the relevant properties of the basic functions $e_\alpha(t)$ directly from their representation as a Laplace inverse integral

$$e_\alpha(t) = \frac{1}{2\pi i} \int_{Br} \mathrm{e}^{\,st}\, \frac{s^{\alpha-1}}{s^\alpha + 1}\, ds\,, \qquad (F.19)$$

in detail for $0 < \alpha \le 2$, without detouring on the general theory of Mittag-Leffler functions in the complex plane. In (F.19) Br denotes a Bromwich path, *i.e.* a line $\Re\,()s = \sigma$ with a value $\sigma \ge 1$ and $\mathrm{Im}\, s$ running from $-\infty$ to $+\infty$.

For transparency reasons, we separately discuss the cases (a) $0 < \alpha < 1$ and (b) $1 < \alpha < 2$, recalling that in the limiting cases $\alpha = 1\,,\, 2$, we know $e_\alpha(t)$ as elementary function, namely $e_1(t) = \mathrm{e}^{-t}$ and $e_2(t) = \cos\, t$. For α not integer the power function s^α is uniquely defined as $s^\alpha = |s|^\alpha\, \mathrm{e}^{i \arg s}$, with $-\pi < \arg s < \pi$, that is in the complex s-plane cut along the negative real axis.

The essential step consists in decomposing $e_\alpha(t)$ into two parts according to $e_\alpha(t) = f_\alpha(t) + g_\alpha(t)$, as indicated below. In case (a) the function $f_\alpha(t)$, in case (b) the function $-f_\alpha(t)$ is *completely monotone*; in both cases $f_\alpha(t)$ tends to zero as t tends to infinity, from above in case (a), from below in case (b). The other part, $g_\alpha(t)$, is identically vanishing in case (a), but of *oscillatory* character with exponentially decreasing amplitude in case (b).

For the oscillatory part we obtain via the residue theorem of complex analysis

$$g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right)\right], \quad \text{if} \quad 1 < \alpha < 2\,. \qquad (F.20)$$

We note that this function exhibits oscillations with circular frequency $\omega(\alpha) = \sin(\pi/\alpha)$ and with an exponentially decaying amplitude with rate $\lambda(\alpha) = |\cos(\pi/\alpha)| = -\cos(\pi/\alpha)$.

For the monotonic part we obtain

$$f_\alpha(t) := \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha(r)\, dr\,, \qquad (F.21)$$

with

$$K_\alpha(r) = -\frac{1}{\pi}\, \mathrm{Im}\left\{ \left. \frac{s^{\alpha-1}}{s^\alpha + 1} \right|_{s\, =\, r\, \mathrm{e}^{i\pi}} \right\} = \frac{1}{\pi}\, \frac{r^{\alpha-1}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\, r^\alpha\, \cos(\alpha\pi) + 1}\,. \qquad (F.22)$$

This function $K_\alpha(r)$ vanishes identically if α is an integer, it is positive for all r if $0 < \alpha < 1$, negative for all r if $1 < \alpha < 2$. In fact in (F.22) the denominator is, for α not integer, always positive being $> (r^\alpha - 1)^2 \ge 0$. Hence $f_\alpha(t)$ has the aforementioned monotonicity properties, decreasing towards zero in case (a), increasing towards zero in case (b). We also note that, in order to satisfy the initial condition $e_\alpha(0^+) = 1$, we find $\int_0^\infty K_\alpha(r)\, dr = 1$ if $0 < \alpha < 1$, $\int_0^\infty K_\alpha(r)\, dr = 1 - 2/\alpha$ if $1 < \alpha < 2$. In Fig. 2 we display the plots of $K_\alpha(r)$, that we denote as the *basic spectral function*, for some values of α in the intervals (a) $0 < \alpha < 1$, (b) $1 < \alpha < 2$.

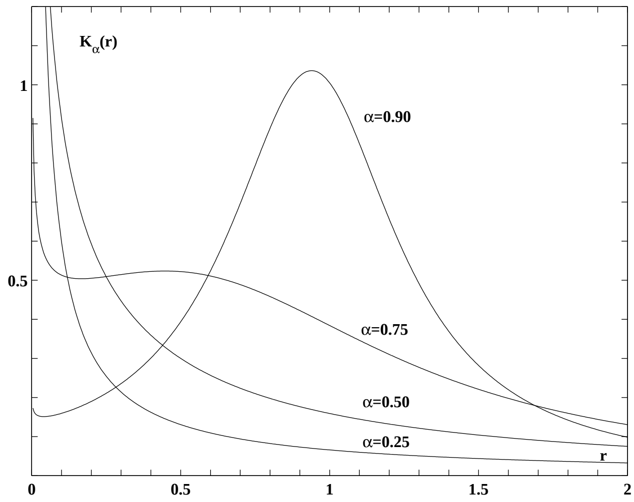

Fig. 2a – Plots of the *basic spectral function* $K_\alpha(r)$ for $0 < \alpha < 1$:
$\alpha = 0.25\,,\quad \alpha = 0.50\,,\quad \alpha = 0.75\,,\quad \alpha 0.90\,.$

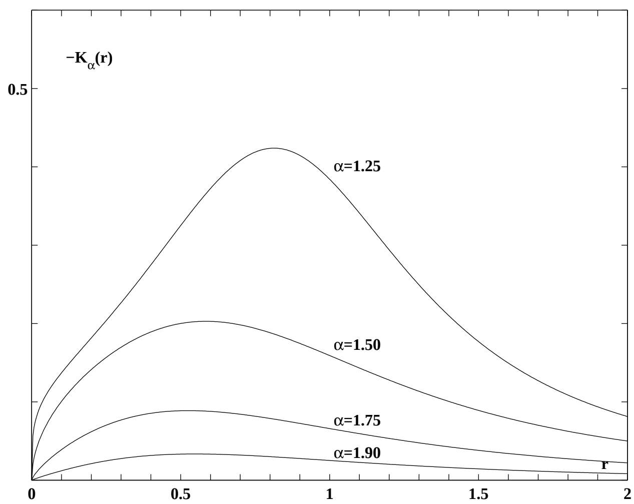

Fig. 1b – Plots of the *basic spectral function* $-K_\alpha(r)$ for $1 < \alpha < 2$:
$\alpha = 1.25\,,\quad \alpha = 1.50\,,\quad \alpha = 1.75\,,\quad \alpha = 1.90\,.$

In addition to the basic fundamental solutions, $u_0(t) = e_\alpha(t)$, we need to compute the impulse-response solutions $u_\delta(t) = -D^1\, e_\alpha(t)$ for cases (a) and (b) and, only in case (b), the second fundamental solution $u_1(t) = J^1\, e_\alpha(t)$.

For this purpose we note that in general it turns out that

$$J^k\, f_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^k(r)\, dr\,, \tag{F.23}$$

with

$$K_\alpha^k(r) := (-1)^k\, r^{-k}\, K_\alpha(r) = \frac{(-1)^k}{\pi}\, \frac{r^{\alpha-1-k}\, \sin(\alpha\pi)}{r^{2\alpha} + 2\, r^\alpha\, \cos(\alpha\pi) + 1}\,, \tag{F.24}$$

where $K_\alpha(r) = K_\alpha^0(r)$, and

$$J^k g_\alpha(t) = \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) - k\frac{\pi}{\alpha}\right]. \tag{F.25}$$

This can be done in direct analogy to the computation of the functions $e_\alpha(t)$, the Laplace transform of $J^k e_\alpha(t)$ being given by (F.17). For the impulse-response solution we note that the effect of the differential operator D^1 is the same as that of the virtual operator J^{-1} .

In conclusion we can resume the solutions for the fractional relaxation and oscillation equations as follows:
(a) $0 < \alpha < 1$,

$$u(t) = c_0\, u_0(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{F.26a}$$

where

$$\begin{cases} u_0(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^0(r)\, dr\,, \\ u_\delta(t) = -\int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^{-1}(r)\, dr\,, \end{cases} \tag{F.27a}$$

with $u_0(0^+) = 1\,,\ u_\delta(0^+) = \infty$;
(b) $1 < \alpha < 2$,

$$u(t) = c_0\, u_0(t) + c_1\, u_1(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{F.26b}$$

where

$$\begin{cases} u_0(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^0(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right)\right], \\ u_1(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^1(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right], \\ u_\delta(t) = -\int_0^\infty \mathrm{e}^{-rt}\, K_\alpha^{-1}(r)\, dr - \frac{2}{\alpha}\, \mathrm{e}^{t\, \cos(\pi/\alpha)}\, \cos\left[t\, \sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right], \end{cases} \tag{F.27b}$$

with $u_0(0^+) = 1$, $u_0'(0^+) = 0$, $u_1(0^+) = 0$, $u_1'(0^+) = 1$, $u_\delta(0^+) = 0$, $u_\delta'(0^+) = +\infty$.

In Figs. 3 we display the plots of the basic fundamental solution for the following cases : (a) $\alpha = 0.25\,, 0.50\,, 0.75\,, 1\,,$ and (b) $\alpha = 1.25\,, 1.50\,, 1.75\,, 2\,,$ obtained from the first formula in (F.27a) and (F.27b), respectively. We have verified that our present results confirm those obtained by Blank (1996) [12] by a numerical calculations and those obtained by Mainardi (1996) [83] by an analytical treatment, valid when α is a rational number, see later. Of particular interest is the case $\alpha = 1/2$ where we recover a well-known formula of the Laplace transform theory, see *e.g.* Doetsch (1974) [35]

$$e_{1/2}(t) := E_{1/2}(-\sqrt{t}) = \mathrm{e}^{\,t}\,\mathrm{erfc}(\sqrt{t}) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s^{1/2}\,(s^{1/2}+1)}\,, \qquad (F.28)$$

where erfc denotes the *complementary error* function defined in (F.4). Explicitly we have

$$E_{1/2}(-\sqrt{t}) = \mathrm{e}^{\,t}\,\frac{2}{\sqrt{\pi}}\int_{\sqrt{t}}^{\infty} \mathrm{e}^{\,-u^2}\,du\,. \qquad (F.29)$$

We now want to point out that in both the cases (a) and (b) (in which α is just not integer) *i.e.* for *fractional relaxation* and *fractional oscillation*, all the fundamental and impulse-response solutions exhibit an *algebraic decay* as $t \to \infty$, as discussed below. This algebraic decay is the most important effect of the non-integer derivative in our equations, which dramatically differs from the *exponential decay* present in the ordinary relaxation and damped-oscillation phenomena.

Let us start with the asymptotic behaviour of $u_0(t)$. To this purpose we first derive an asymptotic series for the function $f_\alpha(t)$, valid for $t \to \infty$. We then consider the spectral representation (F.21-22) and expand the spectral function for small r. Then the Watson lemma yields, see **Problem (F.10)**,

$$f_\alpha(t) = \sum_{n=1}^{N} (-1)^{n-1}\,\frac{t^{-n\alpha}}{\Gamma(1-n\alpha)} + O\left(t^{-(N+1)\alpha}\right)\,, \quad \text{as} \quad t \to \infty\,. \qquad (F.30)$$

We note that this asymptotic expansion coincides with that for $u_0(t) = e_\alpha(t)$, having assumed $0 < \alpha < 2$ ($\alpha \neq 1$). In fact the contribution of $g_\alpha(t)$ is identically zero if $0 < \alpha < 1$ and exponentially small as $t \to \infty$ if $1 < \alpha < 2$.

The asymptotic expansions of the solutions $u_1(t)$ and $u_\delta(t)$ are obtained from (F.30) integrating or differentiating term by term with respect to t. Taking the leading term of the asymptotic expansions, we obtain the asymptotic representations of the solutions $u_0(t)$, $u_1(t)$ and $u_\delta(t)$ as $t \to \infty$,

$$u_0(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad u_1(t) \sim \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,, \quad u_\delta(t) \sim -\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\,, \qquad (F.31)$$

that point out the algebraic decay.

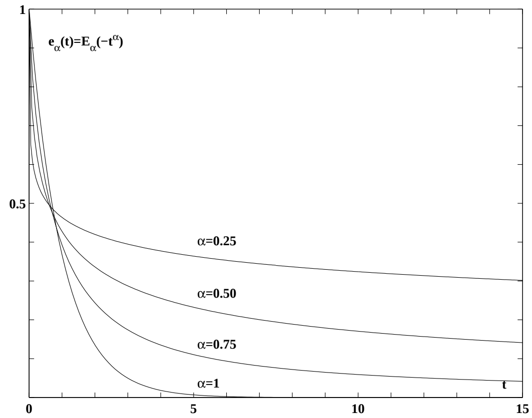

Fig. 3a – Plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $0 < \alpha \le 1$: $\alpha = 0.25\,,\ \ \alpha = 0.50\,,\ \ \alpha = 0.75\,,\ \ \alpha = 1\,.$

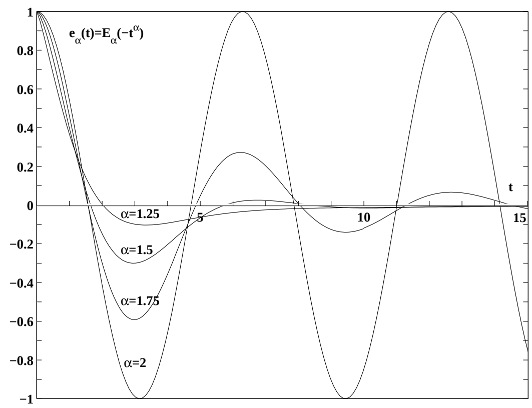

Fig. 3b – Plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $1 < \alpha \le 2$: $\alpha = 1.25\,,\ \ \alpha = 1.50\,,\ \ \alpha = 1.75\,,\ \ \alpha = 2\,.$

We would like to remark the difference between fractional relaxation governed by the Mittag-Leffler type function $E_\alpha(-at^\alpha)$ and stretched relaxation governed by a stretched exponential function $\exp(-bt^\alpha)$ with $\alpha\,, a\,, b > 0$ for $t \ge 0$. A common behaviour is achieved only in a restricted range $0 \le t \ll 1$ where we can have

$$E_\alpha(-at^\alpha) \simeq 1 - \frac{a}{\Gamma(\alpha+1)}\, t^\alpha = 1 - b\, t^\alpha \simeq \mathrm{e}^{-b\, t^\alpha}\ , \text{ if } \quad b = \frac{a}{\Gamma(\alpha+1)}\,. \qquad (F.32)$$

In Fig. 4 we compare, for $\alpha = 0.25\,,\ 0.50\,,\ 0.75$ from top to bottom, $E_\alpha(-t^\alpha)$ (*full* line) with its asymptotic approximations $\exp\left[-t^\alpha/\Gamma(1+\alpha)\right]$ (*dashed* line) valid for short times, see (F.31), and $t^{-\alpha}/\Gamma(1-\alpha)$ (*dotted* line) valid for long times, see (F.30). We have adopted log-log plots in order to better achieve such a comparison and the transition from the stretched exponential to the inverse power-law decay.

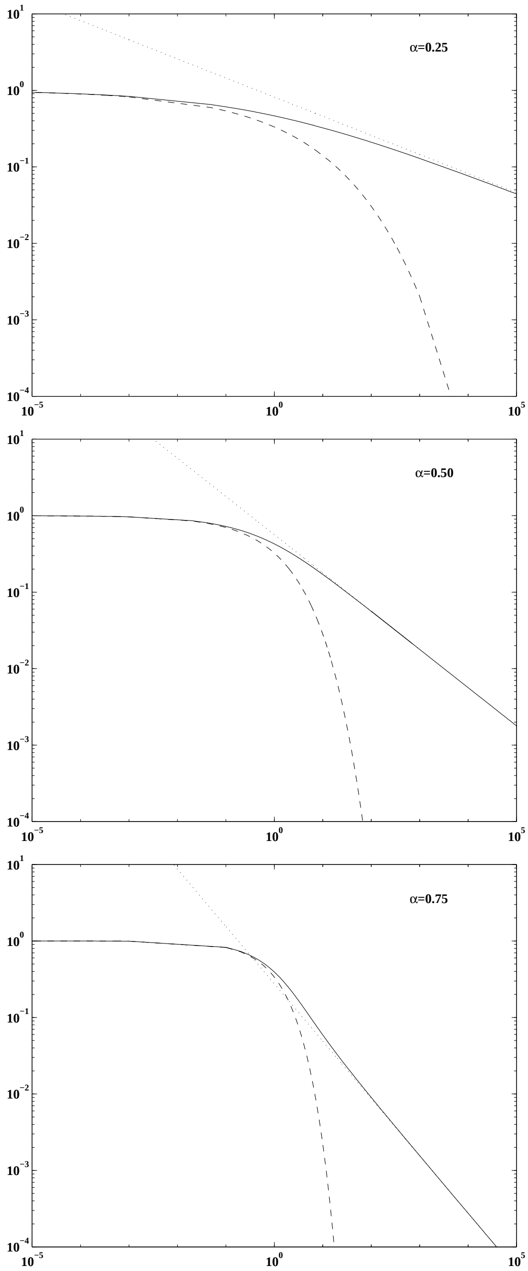

Fig. 4 – Log-log plot of $E_\alpha(-t^\alpha)$ for $\alpha = 0.25, 0.50, 0.75$ in the range $10^{-6} \le t \le 10^{6}$.

In Fig. 5 we show some plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25\,,\ 1.50\,,\ 1.75$ from top to bottom. Here the algebraic decay of the fractional oscillation can be recognized and compared with the two contributions provided by f_α (monotonic behaviour, dotted line) and $g_\alpha(t)$ (exponentially damped oscillation, dashed line).

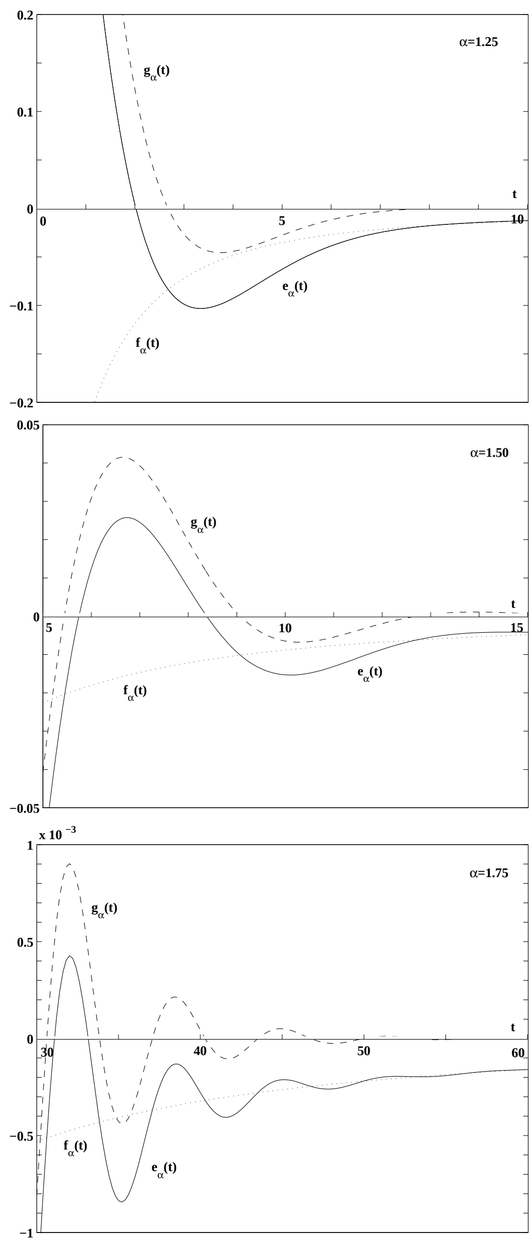

Fig. 5 – Decay of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25, 1.50, 1.75$

The zeros of the solutions of the fractional oscillation equation

Now we find it interesting to carry out some investigations about the zeros of the basic fundamental solution $u_0(t) = e_\alpha(t)$ in the case (b) of fractional oscillations. For the second fundamental solution and the impulse-response solution the analysis of the zeros can be easily carried out analogously.

Recalling the first equation in (F.27b), the required zeros of $e_\alpha(t)$ are the solutions of the equation

$$e_\alpha(t) = f_\alpha(t) + \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right] = 0\,. \qquad (F.33)$$

We first note that the function $e_\alpha(t)$ exhibits an *odd* number of zeros, in that $e_\alpha(0) = 1$, and, for sufficiently large t, $e_\alpha(t)$ turns out to be permanently negative, as shown in (F.31) by the sign of $\Gamma(1-\alpha)$. The smallest zero lies in the first positivity interval of $\cos[t\sin(\pi/\alpha)]$, hence in the interval $0 < t < \pi/[2\,\sin(\pi/\alpha)]$; all other zeros can only lie in the succeeding positivity intervals of $\cos[t\,\sin(\pi/\alpha)]$, in each of these two zeros are present as long as

$$\frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)} \ge |f_\alpha(t)|\,. \qquad (F.34)$$

When t is sufficiently large the zeros are expected to be found approximately from the equation

$$\frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)} \approx \frac{t^{-\alpha}}{|\Gamma(1-\alpha)|}\,, \qquad (F.35)$$

obtained from (F.33) by ignoring the oscillation factor of $g_\alpha(t)$ [see (F.20)] and taking the first term in the asymptotic expansion of $f_\alpha(t)$ [see (F.30)-(F.31)]. As we have shown in [47], such approximate equation turns out to be useful when $\alpha \to 1^+$ and $\alpha \to 2^-$.

For $\alpha \to 1^+$, only one zero is present, which is expected to be very far from the origin in view of the large period of the function $\cos[t\,\sin(\pi/\alpha)]$. In fact, since there is no zero for $\alpha = 1$, and by increasing α more and more zeros arise, we are sure that only one zero exists for α sufficiently close to 1. Putting $\alpha = 1 + \epsilon$ the asymptotic position T_* of this zero can be found from the relation (F.35) in the limit $\epsilon \to 0^+$. Assuming in this limit the first-order approximation, we get

$$T_* \sim \log\left(\frac{2}{\epsilon}\right)\,, \qquad (F.36)$$

which shows that T_* tends to infinity slower than $1/\epsilon$, as $\epsilon \to 0$. For details see [47].

For $\alpha \to 2^-$, there is an increasing number of zeros up to infinity since $e_2(t) = \cos t$ has infinitely many zeros [in $t_n^* = (n + 1/2)\pi$, $n = 0, 1, \ldots$]. Putting now $\alpha = 2 - \delta$

the asymptotic position T_* for the largest zero can be found again from (F.35) in the limit $\delta \to 0^+$. Assuming in this limit the first-order approximation, we get

$$T_* \sim \frac{12}{\pi\,\delta} \log\left(\frac{1}{\delta}\right) . \tag{F.37}$$

For details see [47]. Now, for $\delta \to 0^+$ the length of the positivity intervals of $g_\alpha(t)$ tends to π and, as long as $t \le T_*$, there are two zeros in each positivity interval. Hence, in the limit $\delta \to 0^+$, there is in average one zero per interval of length π , so we expect that $N_* \sim T_*/\pi$.

Remark : For the above considerations we got inspiration from an interesting paper by Wiman (1905) [131], who at the beginning of the XX-th century, after having treated the Mittag-Leffler function in the complex plane, considered the position of the zeros of the function on the negative real axis (without providing any detail). Our expressions of T_* are in disagreement with those by Wiman for numerical factors; however, the results of our numerical studies carried out in [47] confirm and illustrate the validity of our analysis.

Here, we find it interesting to analyse the phenomenon of the transition of the (odd) number of zeros as $1.4 \le \alpha \le 1.8$. For this purpose, in Table I we report the intervals of amplitude $\Delta\alpha = 0.01$ where these transitions occur, and the location T_* (evaluated within a relative error of 0.1%) of the largest zeros found at the two extreme values of the above intervals. We recognize that the transition from 1 to 3 zeros occurs as $1.40 \le \alpha \le 1.41$, that one from 3 to 5 zeros occurs as $1.56 \le \alpha \le 1.57$, and so on. The last transition in the considered range of α is from 15 to 17 zeros, and it just occurs as $1.79 \le \alpha \le 1.80$.

N_*	α	T_*
$1 \div 3$	$1.40 \div 1.41$	$1.730 \div 5.726$
$3 \div 5$	$1.56 \div 1.57$	$8.366 \div 13.48$
$5 \div 7$	$1.64 \div 1.65$	$14.61 \div 20.00$
$7 \div 9$	$1.69 \div 1.70$	$20.80 \div 26.33$
$9 \div 11$	$1.72 \div 1.73$	$27.03 \div 32.83$
$11 \div 13$	$1.75 \div 1.76$	$33.11 \div 38.81$
$13 \div 15$	$1.78 \div 1.79$	$39.49 \div 45.51$
$15 \div 17$	$1.79 \div 1.80$	$45.51 \div 51.46$

Table I

N_* = number of zeros, α = fractional order, T_* location of the largest zero.

Other formulas: summation and integration

For completeness hereafter we exhibit some formulas related to summation and integration of ordinary Mittag-Leffler functions (in one parameter α), referring the interested reader to [31], [107] for their proof and for their generalizations to two parameters.

Concerning summation we outline, see **Problem (F.11)**,

$$E_\alpha(z) = \frac{1}{p}\sum_{h=0}^{p-1} E_{\alpha/p}\left(z^{1/p} e^{i2\pi h/p}\right), \quad p \in \mathbb{N}, \qquad (F.38)$$

from which we derive the *duplication formula*, see also **Problem (F12)**,

$$E_\alpha(z) = \frac{1}{2}\left[E_{\alpha/2}(+z^{1/2}) + E_{\alpha/2}(-z^{1/2})\right]. \qquad (F.39)$$

As an example of this formula we can recover, for $\alpha = 2$, the expressions of cosh z and cos z in terms of two exponential functions.

Concerning integration we outline another interesting *duplication formula*

$$E_{\alpha/2}(-t^{\alpha/2}) = \frac{1}{\sqrt{\pi\, t}}\int_0^\infty \mathrm{e}^{-x^2/(4t)}\, E_\alpha(-x^\alpha)\, dx, \quad x > 0, \quad t > 0. \qquad (F.40)$$

It can be derived by applying a theorem of the Laplace transform theory (known as Efros theorem), see **Ptoblem (F.13)**.

The Mittag-Leffler functions of rational order

Let us now consider the Mittag-Leffler functions of rational order $\alpha = p/q$ with $p, q \in \mathbb{N}$, relatively prime. The relevant functional relations, that we quote from Djrbashian (1966) [31], and Erdélyi (1955) Vol. 3 [36], turn out to be

$$\left(\frac{d}{dz}\right)^p E_p\left(z^p\right) = E_p\left(z^p\right), \qquad (F.41)$$

$$\frac{d^p}{dz^p} E_{p/q}\left(z^{p/q}\right) = E_{p/q}\left(z^{p/q}\right) + \sum_{k=1}^{q-1} \frac{z^{-k\,p/q}}{\Gamma(1-k\,p/q)}, \quad q = 2, 3, \dots, \qquad (F.42)$$

$$E_{p/q}\left(z\right) = \frac{1}{p}\sum_{h=0}^{p-1} E_{1/q}\left(z^{1/p} e^{i2\pi h/p}\right), \qquad (F.43)$$

and

$$E_{1/q}\left(z^{1/q}\right) = \mathrm{e}^z\left[1 + \sum_{k=1}^{q-1} \frac{\gamma(1-k/q, z)}{\Gamma(1-k/q)}\right], \quad q = 2, 3, \dots, \qquad (F.44)$$

where $\gamma(a,z) := \int_0^z \mathrm{e}^{-u}\, u^{a-1}\, du$ denotes the *incomplete gamma function.* defined as

$$\gamma(a,z) := \int_0^z \mathrm{e}^{-u}\, u^{a-1}\, du\,. \tag{F.45}$$

One easily recognizes that the relations (F.41) and (F.42) are immediate consequences of the definition (F.1). To prove the relations (F.43) and (F.44) we refer the reader to the **Problems (F.14), (F.15)**, respectively.

The relation (F.44) shows how the Mittag-Leffler functions of rational order can be expressed in terms of exponentials and incomplete gamma functions. In particular, taking in (F.44) $q = 2$, we now can verify again the relation (F.4). In fact, from (F.44) we obtain

$$E_{1/2}(z^{1/2}) = \mathrm{e}^z \left[1 + \frac{1}{\sqrt{\pi}}\,\gamma(1/2\,,\, z)\right], \tag{F.46}$$

which is equivalent to (F.4) if we use the relation

$$\mathrm{erf}\,(z) = \gamma(1/2, z^2)/\sqrt{\pi}\,, \tag{F.47}$$

see *e.g.* Erdély (1953) Vol. 1 [36], Abramowitz & Stegun (1965) [3].

G. The Wright functions

The *Wright* function, that we denote by $\Phi_{\lambda,\mu}(z)$, $z \in \mathbb{C}$, with the parameters $\lambda > -1$ and $\mu \in \mathbb{C}$, is so named after the British mathematician E. Maitland Wright, who introduced and investigate it between 1933 and 1940 [134, 135, 136, 137, 138]. We note that originally Wright considered such a function restricted to $\lambda \geq 0$ in his paper [134] in connection with his investigations in the asymptotic theory of partitions. Only later, in 1940, he extended to $-1 < \lambda < 0$ [138].

Like for the Mittag-Leffler functions, a description of the most important properties of the Wright functions (with relevant references up to the fifties) can be found in the third volume of the Bateman Project [36], in the chapter $XVIII$ on the *Miscellaneous Functions*. However, probably for a misprint, there λ is restricted to be positive.

Relevant investigations on the Wright functions have been carried out by Stanković, [122], [41], among other authors quoted in Kiryakova's book [68] [pag. 336], and, more recently, by Luchko & Gorenflo (1998) [77], Gorenflo, Luchko & Mainardi (1999, 2000) [45, 46] and Luchko (2000) [76].

The special cases $\lambda = -\nu$, $\mu = 0$ and $\lambda = -\nu$, $\mu = 1 - \nu$ with $0 < \nu < 1$ and z replaced by $-z$ provide the Wright type functions, $F_\nu(z)$ and $M_\nu(z)$, respectively, that have been so denoted and investigated by Mainardi (1994, 1995a, 1995b, 1996a, 1996b, 1997), [79, 80, 81, 82, 83, 84]. Since these functions are of special interest for us, we shall later return to them and to present a detailed analysis, see also Gorenflo, Luchko & Mainardi (1999, 2000) [45, 46]. We refer to them as the auxiliary functions of the Wright type.

The series representation of the Wright function

The Wright function is defined by the power series convergent in the whole complex plane,

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{G.1}$$

The case $\lambda = 0$ is trivial since the Wright function is reduced to the exponential function with the constant factor $1/\Gamma(\mu)$, which turns out to vanish identically for $\mu = -n$, $n = 0, 1, \ldots$ In general it is proved that the Wright function for $\lambda > -1$ and $\mu \in \mathbb{C}$ ($\mu \neq -n$, $n = 0, 1, \ldots$ if $\lambda = 0$) is an entire function of finite order ρ and type σ given by, see *e.g.* Gorenflo, Luchko & Mainardi (1999) [45] and **Problems (G.1)**,

$$\rho = \frac{1}{1+\lambda}, \quad \sigma = (1+\lambda)\,|\lambda|^{\lambda/(1+\lambda)}. \tag{G.2}$$

In particular, the Wright function turns out to be of *exponential type* if $\lambda \geq 0$.

The Wright integral representation and asymptotic expansions

The integral representation of the Wright function reads

$$\Phi_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\zeta + z\zeta^{-\lambda}} \frac{d\zeta}{\zeta^{\mu}}\,, \quad \lambda > -1\,, \quad \mu \in \mathbb{C}\,, \quad z \in \mathbb{C}\,. \qquad (G.3)$$

Here Ha denotes an arbitrary Hankel path, namely a contour consisting of pieces of the two rays $\arg\zeta = \pm\phi$ extending to infinity, and of the circular arc $\zeta = \epsilon\, \mathrm{e}^{i\theta}$, $|\theta| \le \phi$, with $\phi \in (\pi/2, \pi)$, and $\epsilon > 0$, arbitrary. The identity between the integral and series representations is obtained by using the Hankel representation of the reciprocal of the gamma function, see **Problem (G.2)**.

The complete picture of the asymptotic behaviour of the Wright function for large values of z was given by Wright himself by using the method of steepest descent on the integral representation (G.3). In particular the papers [135, 136] were devoted to the case $\lambda > 0$ and the paper [138] to the case $-1 < \lambda < 0$. Wright's results have been summarized by Gorenflo, Luchko & Mainardi (1999, 2000) [45, 46]. Recently, Wong & Zhao (1999a, 1999b) [132, 133] have provided a detailed asymptotic analysis of the Wright function in the cases $\lambda > 0$ and $-1 < \lambda < 0$ respectively, achieving a uniform "exponentially improved" expansion with a smooth transition across the Stokes lines. The asymptotics of zeros of the Wright function has been investigated by Luchko (2000) [76].

Here we limit ourselves to recall Wright's result [138] in the case $\lambda = -\nu \in (-1, 0)$, $\mu > 0$ where the following asymptotic expansion is valid in a suitable sector about the negative real axis

$$\Phi_{-\nu,\mu}(z) = Y^{\,1/2-\mu}\, \mathrm{e}^{-Y} \left(\sum_{m=0}^{M-1} A_m\, Y^{-m} + O(|Y|^{-M}) \right), \quad |z| \to \infty\,, \qquad (G.4)$$

with $Y = (1-\nu)\,(-\nu^{\nu}\, z)^{1/(1-\nu)}$, where the A_m are certain real numbers.

The Wright functions as generalization of the Bessel functions

For $\lambda = 1$ and $\mu = \nu + 1$ the Wright function turns out to be related to the well known Bessel functions J_ν and I_ν by the following identity, see **Problem (G.3)**,

$$(z/2)^{\nu}\, \Phi_{1,\nu+1}\left(\mp z^2/4\right) = \begin{cases} J_\nu(z) \\ I_\nu(z) \end{cases}. \qquad (G.5)$$

In view of this property some authors refer to the Wright function as the *Wright generalized Bessel function* (misnamed also as the *Bessel-Maitland function*) and introduce the notation

$$J_\nu^{(\lambda)}(z) := \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\lambda n + \nu + 1)}\,; \quad J_\nu^{(1)}(z) := J_\nu(z)\,. \qquad (G.6)$$

As a matter of fact, the Wright function appears as the natural generalization of the entire function known as *Bessel - Clifford function*, see *e.g.* Kiryakova (1994) [68], pag 336, and referred by Tricomi, see *e.g.* Tricomi (1960) [124], Gatteschi (1973) [42] as the *uniform Bessel function*

$$T_\nu(z) := z^{-\nu/2}\, J_\nu(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,\Gamma(n+\nu+1)} = \Phi_{1,\nu+1}(-z)\,. \qquad (G.7)$$

Some of the properties which the Wright functions share with the popular Bessel functions were enumerated by Wright himself. Hereafter, we quote two relevant relations from the Bateman Project [36], which can easily be derived from (G.1) or (G.3), see **Problem (G.4)**:

$$\lambda z\, \Phi_{\lambda,\lambda+\mu}(z) = \Phi_{\lambda,\mu-1}(z) + (1-\mu)\, \Phi_{\lambda,\mu}(z)\,, \qquad (G.8)$$

$$\frac{d}{dz}\, \Phi_{\lambda,\mu}(z) = \Phi_{\lambda,\lambda+\mu}(z)\,. \qquad (G.9)$$

The auxiliary functions $F_\nu(z)$ and $M_\nu(z)$ of the Wright type

In our treatment of the time fractional diffusion wave equation we findit convenient to introduce two *auxiliary functions* $F_\nu(z)$ and $M_\nu(z)$, where z is a complex variable and ν a real parameter $0 < \nu < 1$. Both functions turn out to be analytic in the whole complex plane, *i.e.* they are entire functions. Their respective *integral representations* read,

$$F_\nu(z) := \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\zeta - z\zeta^\nu}\, d\zeta\,, \quad 0<\nu<1\,, \quad z \in \mathbb{C}\,, \qquad (G.10)$$

$$M_\nu(z) := \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\zeta - z\zeta^\nu}\, \frac{d\zeta}{\zeta^{1-\nu}}\,, \quad 0<\nu<1\,, \quad z \in \mathbb{C}\,. \qquad (G.11)$$

From a comparison of (G.10-11) with (G.3) we easily recognize that

$$F_\nu(z) = \Phi_{-\nu,0}(-z)\,, \qquad (G.12)$$

and

$$M_\nu(z) = \Phi_{-\nu,1-\nu}(-z)\,. \qquad (G.13)$$

From (G.8) and (G.12-13) we find the relation, see **Problem (G.5)**,

$$F_\nu(z) = \nu\, z\, M_\nu(z)\,. \qquad (G.14)$$

This relation can be obtained directly from (G.10-11) via an integration by parts,

$$\int_{Ha} \mathrm{e}^{\,\zeta - z\zeta^\nu}\, \frac{d\zeta}{\zeta^{1-\nu}} = \int_{Ha} \mathrm{e}^{\,\zeta} \left(-\frac{1}{\nu z}\, \frac{d}{d\zeta} \mathrm{e}^{\,-z\zeta^\nu}\right) d\zeta = \frac{1}{\nu z} \int_{Ha} \mathrm{e}^{\,\zeta - z\zeta^\nu}\, d\zeta\,.$$

The *series representations* for our auxiliary functions turn out to be respectively, see **Problem (G.6)**,

$$F_\nu(z) := \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\,\Gamma(-\beta n)} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^n}{n!}\,\Gamma(\nu n+1)\,\sin(\pi\nu n)\,, \qquad (G.15)$$

$$M_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n+(1-\nu)]} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\,\Gamma(\nu n)\,\sin(\pi\nu n)\,. \qquad (G.16)$$

The series at the R.H.S. have been obtained by using the well-known reflection formula for the Gamma function $\Gamma(\zeta)\,\Gamma(1-\zeta) = \pi/\sin \pi\zeta$. Furthermore we note that $F_\nu(0) = 0$, $M_\nu(0) = 1/\Gamma(1-\nu)$ and that the relation (G.14) can be derived also from (G.15-16).

Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of ν . Mainardi & Tomirotti (1995) [89] have shown that for $\nu = 1/q$, with an integer $q \ge 2$, the auxiliary function $M_\nu(z)$ can be expressed as a sum of $(q-1)$ simpler entire functions, namely, see **Problem (G.7)**,

$$M_{1/q}(z) = \frac{1}{\pi} \sum_{h=1}^{q-1} c(h,q)\,G(z;h,q) \qquad (G.17)$$

with

$$c(h,q) = (-1)^{h-1}\,\Gamma(h/q)\,\sin(\pi h/q)\,, \qquad (G.18)$$

$$G(z;h,q) = \sum_{m=0}^{\infty} (-1)^{m(q+1)} \left(\frac{h}{q}\right)_m \frac{z^{qm+h-1}}{(qm+h-1)!}\,. \qquad (G.19)$$

Here $(a)_m$, $m = 0,1,2,\ldots$ denotes Pochhammer's symbol

$$(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\ldots(a+m-1)\,.$$

We note that $(-1)^{m(q+1)}$ is equal to $(-1)^m$ for q even and $+1$ for q odd. In the particular cases $q = 2$, $q = 3$ we find, see **Problems (G.8), (G.9)**, respectively

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\,z^2/4\right)\,, \qquad (G.20)$$

$$M_{1/3}(z) = 3^{2/3}\,\mathrm{Ai}\left(z/3^{1/3}\right)\,, \qquad (G.21)$$

where Ai denotes the *Airy function*, see *e.g.* [3].

Furthermore it can be proved that $M_{1/q}(z)$ (for integer ≥ 2) satisfies the differential equation of order $q-1$, see **Problem (G.10)**,

$$\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q} z\, M_{1/q}(z) = 0\,, \qquad (G.22)$$

subjected to the $q-1$ initial conditions at $z=0$, derived from the series expansion in (G.17-19),

$$M_{1/q}^{(h)}(0) = \frac{(-1)^h}{\pi}\, \Gamma[(h+1)/q]\, \sin[\pi\,(h+1)/q]\,, \quad h = 0,\, 1,\, \dots\, q-2\,. \qquad (G.23)$$

We note that, for $q \geq 4$, Eq. (G.22) is akin to the *hyper-Airy* differential equation of order $q-1$, see *e.g.* Bender & Orszag (1987)[11]. Consequently, the function $M_\nu(z)$ is a generalization of the hyper-Airy function. In the limiting case $\nu = 1$ we get $M_1(z) = \delta(z-1)$, *i.e.* the M function degenerates into a generalized function of Dirac type.

From our purposes (time-fractional diffusion processes) it is relevant to consider the M_ν function for a positive (real) argument, that will be denoted by r. Later, by using its Laplace transform with the Bernstein theorem, we shall prove that $M_\nu(r) > 0$ for $r > 0$.

In Fig. 6 we compare in $0 \leq r \leq 4$ the plots of $M_\nu(r)$ for some rational values of $\nu \neq 1/2$ with the plot for $\nu = 1/2$ corresponding to the (right-sided) Gaussian. We note that for $0 < \nu \leq 1/2$ the function is monotonically decreasing, while for $1/2 < \nu < 1$ it exhibits a maximum whose position tends to $r = 1$ as $\nu \to 1^-$ (consistently with $M_1(r) = \delta(r-1)$).

The asymptotic representation of $M_\nu(r)$, as $r \to \infty$ can be obtained by using the ordinary saddle-point method. Choosing as a variable r/ν rather than r the computation is easier and yields, see Mainardi & Tomirotti (1995) [89] and **Problem (G.11)**,

$$M_\nu(r/\nu) \sim a(\nu)\, r^{(\nu-1/2)/(1-\nu)} \exp\left[-b(\nu)\, r^{(1/(1-\nu)}\right]\,, \quad r \to +\infty\,, \qquad (G.24)$$

where $a(\nu) = 1/\sqrt{2\pi\,(1-\nu)} > 0$, and $b(\nu) = (1-\nu)/\nu > 0$.

The above asymptotic representation is consistent with the first term of the asymptotic expansion (G.4) obtained by Wright for $\Phi_{-\nu,\mu}(-r)$. In fact, taking $\mu = 1-\nu$ so $1/2 - \mu = \nu - 1/2$, we obtain

$$M_\nu(r) \sim A_0\, Y^{\,\nu-1/2} \exp\,(-Y)\,, \quad r \to \infty\,, \qquad (G.25)$$

where

$$A_0 = \frac{1}{\sqrt{2\pi}\,(1-\nu)^\nu\, \nu^{2\nu-1}}\,, \quad Y = (1-\nu)\,(\nu^\nu\, r)^{1/(1-\nu)}\,. \qquad (G.26)$$

Because of the above exponential decay, any moment of order $\delta > -1$ for $M_\nu(r)$ is finite. In fact, see **Problem (G.12)**,

$$\int_0^\infty r^{\,\delta}\, M_\nu(r)\, dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}\,, \quad \delta > -1\,, \quad 0 < \nu < 1\,. \qquad (G.27)$$

In particular we get the normalization property in $\mathbb{R}^+$, $\int_0^\infty M_\nu(r)\, dr = 1$.

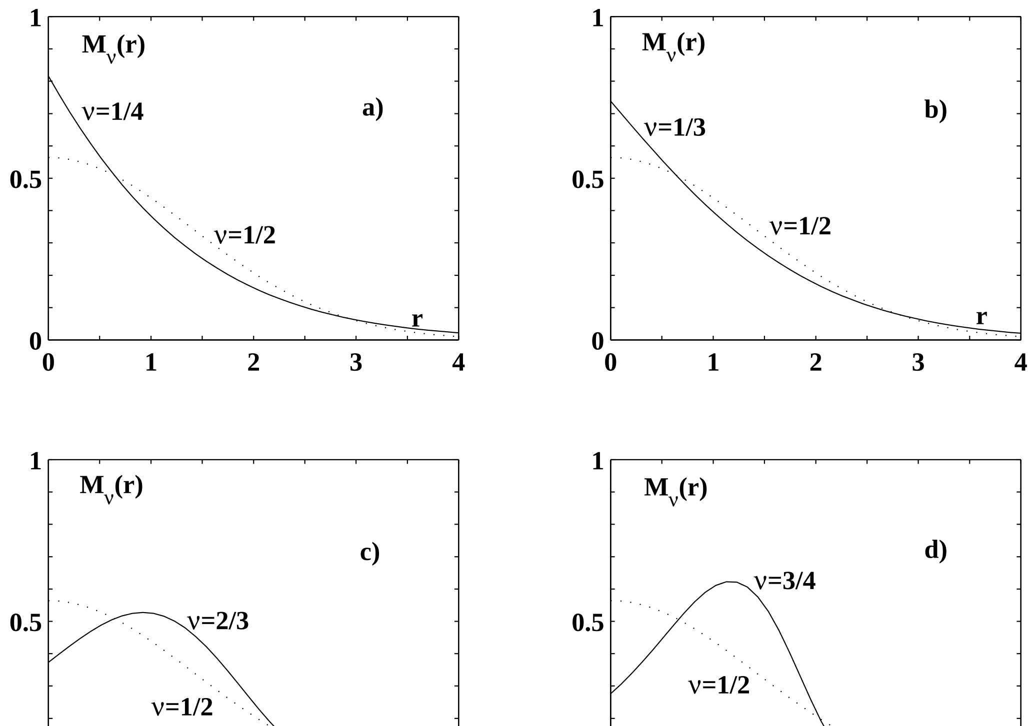

Fig. 6 – Comparison of $M_\nu(r)$ (continuous line) with $M_{1/2}(r)$ (dashed line) in $0 \le r \le 4$: (a) $\nu = 1/4$, (b) $\nu = 1/3$, (c) $\nu = 2/3$, (d) $\nu = 3/4$.

In the Figs. 7a and 7b we show how the plots of the same functions appear after adopting a logarithmic scale for the ordinate. So doing we point out the rate of (exponential) decay versus the parameter ν as it can be deduced from the asymptotic representation (G.26)-(G.27) as $r \to \infty$.

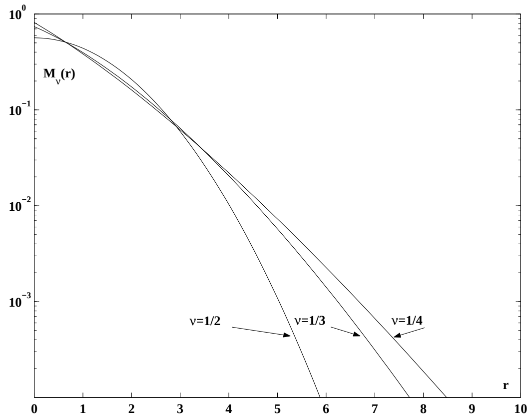

Fig. 7a – Lin-log plot of $M_\nu(r)$ in $0 \le r \le 10$: $\nu = 1/4\,,\ 1/3\,,\ 1/2\,.$

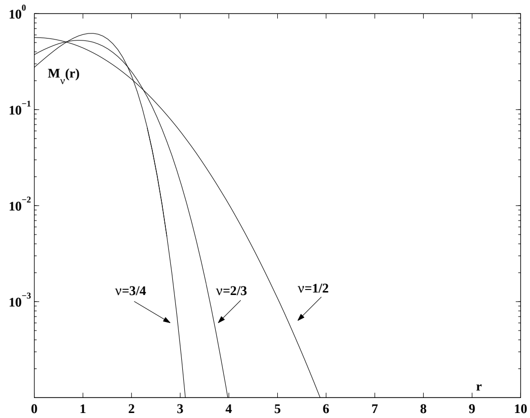

Fig. 7b – Lin-log plot of $M_\nu(r)$ in $0 \le r \le 10$: $\nu = 1/2\,,\ 2/3\,,\ 3/4\,.$

Similarily, we can compute any moment of order $\delta > -1$ of the generic function $\Phi_{-\nu,\mu}(-r)$ in view of its exponential decay (G.4), obtaining

$$\int_0^\infty r^{\,\delta}\, \Phi_{-\nu,\mu}(-r)\, dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+\nu+\mu)}\,, \quad \delta > -1\,, \quad 0 < \nu < 1\,, \quad \mu > 0\,. \quad (G.28)$$

We also quote an interesting formula derived by Stankovič (1970) [122], see also **Problem (G.13)**, which provides a relation between the Whittaker function $W_{-1/2,1/6}$ and the Wright function $\Phi_{-2/3,0} = F_{2/3}$,

$$F_{2/3}\left(x^{-2/3}\right) = -\frac{1}{2\sqrt{3\pi}}\exp\left(-\frac{2}{27x^2}\right) W_{-1/2,1/6}\left(-\frac{4}{27x}\right) . \qquad (G.29)$$

We recall that the generic Whittaker function $W_{\lambda,\mu}(x)$ satisfies the differential equation, see *e.g.* [3]

$$\frac{d^2}{dx^2}W_{\lambda,\mu}(x) + \left(-\frac{1}{4}+\frac{\lambda}{x}+\frac{\mu^2}{4x^2}\right) W_{\lambda,\mu}(x) = 0\,, \quad \lambda,\mu \in \mathbb{R}< \ . \qquad (G.30)$$

Laplace transform pairs related to the Wright function

Let us now consider some Laplace transform pairs related to the Wright functions. We continue to denote by r a positive variable.

In the case $\lambda > 0$ the Wright function is an entire function of order less than 1 and consequently, being of exponential type, its Laplace transform can be obtained by transforming term-by-term its Taylor expansion (G.1) in the origin, see *e.g.* Doetsch (1974) [35], Th. ??. As a result we get, see **Problem (G.14)**,

$$\Phi_{\lambda,\mu}(\pm r) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}\sum_{k=0}^{\infty}\frac{(\pm s^{-1})^k}{\Gamma(\lambda k+\mu)} = \frac{1}{s}E_{\lambda,\mu}(\pm s^{-1})\,, \quad \lambda > 0\,, \quad \mu \in \mathbb{C}\,, \qquad (G.31)$$

with $0 < \epsilon < |s|$, ϵ arbitrarily small. Here $E_{\alpha,\beta}(z)$ denotes the generalized Mittag-Leffler function (F.5). In this case the resulting Laplace transform turns out to be analytic for $s \neq 0$, vanishing at infinity and exhibiting an essential singularity at $s = 0$.

For $-1 < \lambda < 0$ the just applied method cannot be used since then the Wright function is an entire function of order greater than one. In this case, setting $\nu = -\lambda$, the existence of the Laplace transform of the function $\Phi_{-\nu,\mu}(-t),\ t > 0$, follows from (G.4), which says us that the function $\Phi_{-\nu,\mu}(z)$ is exponentially small for large z in a sector of the plane containing the negative real semi-axis. To get the transform in this case we can use the idea given in Mainardi (1997) [84], see also **Problem (G.15)**, based on the integral representation (G.3). We have

$$\Phi_{-\nu,\mu}(-r) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{2\pi i}\int_{Ha}\frac{\mathrm{e}^{\zeta}\zeta^{-\mu}}{s+\zeta^{\nu}}\,d\zeta = E_{\nu,\mu+\nu}(-s)\,, \qquad (G.32)$$

where we have used the integral representation (F.10) of the generalized Mittag-Leffler function.

The relation (G.32) was given in Djrbashian and Bagian (1975) [33] (see also Djrbashian (1993) [32]) in the case $\mu \geq 0$ as a representation of the generalized Mittag-Leffler function in the whole complex plane as a Laplace integral of an entire function but without identifying this function as the Wright function. They also gave (in slightly different notations) the more general representation

$$E_{\alpha_2,\beta_2}(z) = \int_0^\infty E_{\alpha_1,\beta_1}(zr^{\alpha_1})\, r^{\beta_1 - 1}\, \Phi_{-\nu,\gamma}(-r)\, dr\,, \qquad (G.33)$$

with $0 < \alpha_2 < \alpha_1\,,\ \beta_1\,,\, \beta_2 > 0\,$, and $0 < \nu = -\alpha_2/\alpha_1 < 1\,,\ \gamma = \beta_2 - \beta_1\, \alpha_2/\alpha_1\,$.
An important particular case of the Laplace transform pair (G.32) is given for $\mu = 1 - \nu$ to yield, see also Mainardi (1997) [84],

$$M_\nu(r) \ \stackrel{\mathcal{L}}{\leftrightarrow}\ E_\nu(-s)\,, \quad 0 < \nu < 1\,. \qquad (G.34)$$

As a further particular case we recover the well-known Laplace transform pair, see *e.g.* Doetsch (1974) [35],

$$M_{1/2}(r) = \frac{1}{\sqrt{\pi}} \exp\left(-\, r^2/4\right) \ \stackrel{\mathcal{L}}{\leftrightarrow}\ E_{1/2}(-s) := \exp(s^2)\,\mathrm{erfc}\,(s)\,. \qquad (G.35)$$

We also note that, transforming term-by-term the Taylor series of $M_\nu(r)$ (not being of exponential order) yields a series of negative powers of s, which represents the asymptotic expansion of $E_\nu(-s)$ as $s \to \infty$ in a sector around the positive real semi-axis.
Using the relation

$$\int_0^\infty r^n\, f(r)\, dr = \lim_{s\to 0}(-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(r); s\}\,,$$

the Laplace transform pair (G.32) and the series representation of the generalized Mittag-Leffler function (F.5) we can compute all the moments of integer order for the Wright function $\Phi_{-\nu,\mu}(-r)$ with $0 < \nu < 1$ in $\mathbb{R}^+$:

$$\int_0^\infty r^n \Phi_{-\nu,\mu}(-r)\, dr = \frac{n!}{\Gamma(\nu n + \mu + \nu)}\,, \quad n \in \mathbb{N}_0\,. \qquad (G.36)$$

This formula is consistent with the more general formula (G.28) valid when the moments are of arbitrary order $\delta > -1\,$.
We can now obtain other Laplace transform pairs related to our auxiliary functions. Indeed, following Mainardi [84] and using the integral representations (G.10-11), see **Problem (G.15)** we get

$$\frac{1}{r}\, F_\nu\left(cr^{-\nu}\right) = \frac{c\nu}{r^{\nu+1}}\, M_\nu\left(cr^{-\nu}\right) \ \stackrel{\mathcal{L}}{\leftrightarrow}\ \exp(-cs^\nu)\,, \quad 0 < \nu < 1,\ c > 0\,. \qquad (G.37)$$

The Laplace inversion in Eq. (G.37) was properly carried out by Pollard (1948) [109] (based on a formal result by Humbert (1945)[60]). and by Mikusiński (1959) [95]. A formal series inversion was carried out by Buchen & Mainardi (1975) [15], albeit unaware of the previous results, see **Problem (G. 16)**.

By applying the formula for differentiation of the image of the Laplace transform to Eq. (G.37), we get a Laplace transform pair useful for our further discussions, namely

$$\frac{1}{r^\nu}\, M_\nu(cr^{-\nu}) \stackrel{\mathcal{L}}{\leftrightarrow} s^{\nu-1}\exp(-cs^\nu)\,, \quad 0<\nu<1\,, \quad c>0\,. \qquad (G.38)$$

As particular cases of Eqs. (G.37)-(G.38), we recover the well-known pairs, see *e.g.* Doetsch (1974) [35] ,

$$\frac{1}{2r^{3/2}} M_{1/2}(1/r^{1/2})) = \frac{1}{2\sqrt{\pi}}\, r^{-3/2}\exp\left(-1/(4r^2)\right) \stackrel{\mathcal{L}}{\leftrightarrow} \exp\left(-s^{1/2}\right)\,, \qquad (G.39)$$

$$\frac{1}{r^{1/2}} M_{1/2}(1/r^{1/2})) = \frac{1}{\sqrt{\pi}}\, r^{-1/2}\exp\left(-1/(4r^2)\right) \stackrel{\mathcal{L}}{\leftrightarrow} s^{-1/2}\exp\left(-s^{1/2}\right)\,. \qquad (G.40)$$

More generally, using the same method as in (G.37), we get see Stanković (1970) [122] and **Problem (G.17)**, the pair

$$r^{\mu-1}\Phi_{-\nu,\mu}(-cr^{-\nu}) \stackrel{\mathcal{L}}{\leftrightarrow} s^{-\mu}\exp(-cs^\nu)\,, \quad 0<\nu<1\,, \quad c>0\,. \qquad (G.41)$$

Stanković (1970) [122] also gave some other pairs related to the Wright function including:

$$r^{\mu/2-1}\,\Phi_{-\nu,\mu}\left(-r^{-\nu/2}\right) \stackrel{\mathcal{L}}{\leftrightarrow} \frac{\sqrt{\pi}}{2^\mu}\, s^{-\mu/2}\,\Phi_{-\nu/2,(\mu+1)/2}\left(-2^\nu s^{\nu/2}\right)\,, \qquad (G.42)$$

with $0<\nu<1$, see **Problem (G.18)**, and

$$r^{-\mu}\exp\left(-r^\nu\cos(\nu\pi)\right)\,\sin(\mu\pi+r^\nu\,\sin(\nu\pi)) \div \pi s^{\mu-1}\,\Phi_{-\nu,\mu}\left(-s^{-\nu}\right)\,, \qquad (G.43)$$

with $0<\nu<1$ and $\mu<1$, see **Problem (G.19)**.

References

[1] N.H. Abel, Solution de quelques problèmes à l'aide d'intégrales définie [Norwegian], *Magazin for Naturvidenskaberne*, Aargang 1, Bind 2, Christiana 1823. French translation in *Oeuvres Complètes*, Vol I, pp. 11-18. Nouvelle èdition par L. Sylow et S. Lie, 1881.

[2] N.H. Abel, Aufloesung einer mechanischen Aufgabe, *Journal für die reine und angewandte Mathematik* (Crelle), Vol. I (1826), pp. 153-157.

[3] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York 1965.

[4] R.P. Agarwal, A propos d'une note de M. Pierre Humbert, *C.R. Acad. Sci. Paris* **236** (1953) 2031–2032.

[5] Yu.I. Babenko, *Heat and Mass Transfer*, Chimia, Leningrad 1986. [in Russian]

[6] R.T. Baillie and M.L. King (Editors), Fractional Differencing and Long Memory Processes, *Journal of Econometrics* **73** No 1 (1996) 1-324.

[7] A.V. Balakrishnan, Representation of abstract Riesz potentials of the elliptic type, *Bull. Amer. Math. Soc.* **64** No 5 (1958) 266–289.

[8] A.V. Balakrishnan, Operational calculus for infinitesimal generators of semi-groups, *Trans. Amer. Math. Soc.* **91** No 2 (1959) 330-353.

[9] A.V. Balakrishnan, Fractional powers of closed operators and the semi-groups generated by them, *Pacific J. Math.* **10** No 2 (1960) 419-437.

[10] J.H. Barret, Differential equations of non-integer order, *Canad. J. Math.* **6** (1954) 529–541.

[11] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, Singapore 1987, Ch 3.

[12] L. Blank, Numerical treatment of differential equations of fractional order, MCCM Numerical Analysis Report No. 287, The University of Manchester 1996. [`www.ma.man.ac.uk/MCCM/MCCM.html`]

[13] S. Bochner, Diffusion equation and stochastic processes, *Proc. Nat. Acad. Sciences, USA* **35** (1949) 368–370.

[14] B.L.J. Braaksma, *Proc. Second Berkelery Symp. Math. Stat. Prob., 1950* Asymptotic expansions and analytical continuations for a class of Barnes-integrals, *Compositio Mathematica* **15** (1962) 239–341.

[15] P.W. Buchen and F. Mainardi, Asymptotic expansions for transient viscoelastic waves, *Journal de Mécanique* **14** (1975) 597-608.

[16] P. Butzer and U. Westphal, Introduction to fractional calculus, in: H. Hilfer (Editor), *Fractional Calculus, Applications in Physics*, World Scientific, Singapore 2000, pp. 1 - 85.

[17] M. Caputo, Linear models of dissipation whose Q is almost frequency independent: Part II., *Geophys. J. R. Astr. Soc.* **13** (1967) 529-539.

[18] M. Caputo, *Elasticità e Dissipazione*, Zanichelli, Bologna, 1969. [in Italian]

[19] M. Caputo, Vibrations of an infinite viscoelastic layer with a dissipative memory, *J. Acoust. Soc. Am.* **56** No 3 (1974) 897-904.

[20] M. Caputo, Which is the correct stress strain relation for the anelasticity of the earth's interior? *Geophys. J. R. Astr. Soc.* **58** (1979) 432-435.

[21] M. Caputo, Generalized rheology and geophysical consequences, *Tectonophysics* **116** (1985) 163-172.

[22] M. Caputo, The rheology of an anelastic medium studied by means of the observation of the splitting of its eigenfrequencies, *J. Acoust. Soc. Am.* **86** No 5 (1989) 1984-1989.

[23] M. Caputo, Modern rheology and electric induction: multivalued index of refraction, splitting of eigenvalues and fatigues, *Ann. Geofisica* **39** No 5 (1996) 941-966.

[24] M. Caputo, Models of flux in porous media with memory, *Water Resources Research* **36** No 3 (2000) 693-705.

[25] M. Caputo, Distributed order differential equations modelling dielectric induction and diffusion, *Fractional Calculus and Applied Analysis* **4** (2001) 421-44.

[26] M. Caputo and F. Mainardi, A new dissipation model based on memory mechanism, *Pure and Appl. Geophys. (PAGEOPH)* **91** (1971) 134-147.

[27] M. Caputo and F. Mainardi, Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento* (Ser. II) **1** (1971) 161–198.

[28] A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien, 1997.

[29] K.S. Cole, Electrical conductance of biological systems, in: *Proc. Symp. Quant. Biol.* (Cold Spring Harbor, New York, 1933) 107-116.

[30] H.T. Davis, *The Theory of Linear Operators* (The Principia Press, Bloomington, Indiana, 1936).

[31] M.M. Djrbashian, *Integral Transforms and Representations of Functions in the Complex Plane*, Nauka, Moscow 1966. [in Russian]

[32] M.M. Djrbashian, *Harmonic Analysis and Boundary Value Problems in the Complex Domain*, Birkhäuser Verlag, Basel 1993.

[33] M. M. Djrbashian and R. A. Bagian, On integral representations and measures associated with Mittag-Leffler type functions, *Izv. Akad. Nauk Armjanskvy SSR, Matematika* **10** (1975) 483-508. [In Russian]

[34] M.M. Djrbashian and A.B. Nersesian, Fractional derivatives and the Cauchy problem for differential equations of fractional order, *Izv. Acad. Nauk Armjanskvy SSR, Matematika* **3** No 1 (1968) 3–29. [In Russian]

[35] G. Doetsch, *Introduction to the Theory and Applications of the Laplace Transformation* (Springer Verlag, Berlin, 1974).

[36] A. Erdélyi (Editor) *Higher Transcendental Functions*, Bateman Project, Vols. 1-3, McGraw-Hill, New York 1953-1955.

[37] A. Erdélyi (Editor) *Tables of Integral Transforms*, Bateman Project, Vols. 1-2, McGraw-Hill, New York, 1953-1955.

[38] W. Feller, On a generalization of Marcel Riesz' potentials and the semi-groups generated by them, *Meddelanden Lunds Universitets Matematiska Seminarium* (Comm. Sém. Mathém. Université de Lund), Tome suppl. dédié a M. Riesz, Lund (1952), 73-81.

[39] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, 2-nd edn. Wiley, New York 1971. [1-st edn. 1966]

[40] C. Fox, The G and H functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* **98** (1961) 395-429.

[41] Lj. Gajić and B. Stanković, Some properties of Wright's function, *Publ. de l'Institut Mathèmatique, Beograd, Nouvelle Sèr.* **20**, No 34 (1976), 91-98.

[42] L. Gatteschi, *Funzioni Speciali*, UTET, Torino 1973, pp. 196-197.

[43] I.M. Gel'fand and G.E. Shilov, *Generalized Functions*, Vol. 1, Academic Press, New York 1964. [English translation from the Russian (Nauka, Moscow, 1959)]

[44] R. Gorenflo, Fractional calculus: some numerical methods, in: A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien, 1997, pp. 277-290. [Reprinted in `www.fracalmo.org`]

[45] R. Gorenflo, Yu. Luchko and F. Mainardi, Analytical properties and applications of the Wright function, *Fractional Calculus and Applied Analysis* **2** No 4 (1999) 383-414.

[46] R. Gorenflo, Yu. Luchko and F. Mainardi, Wright functions as scale-invariant solutions of the diffusion-wave equation, *J. Computational and Appl. Mathematics* **118** No 1-2 (2000) 175-191.

[47] R. Gorenflo and F. Mainardi, Fractional oscillations and Mittag-Leffler functions, Pre-print A-14/96, Fachbereich Mathematik und Informatik, Freie Universität, Berlin, 1996. [`www.math.fu-berlin.de/publ/index.html`]

[48] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in: A. Carpinteri and F. Mainardi (Editors), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien 1997, pp. 223–276. [Reprinted in `www.fracalmo.org`]

[49] R. Gorenflo and F. Mainardi, Random walk models for space-fractional diffusion processes, *Fractional Calculus and Applied Analysis* **1** No 2 (1998) 167–191.

[50] R. Gorenflo and F. Mainardi, Approximation of Lévy-Feller diffusion by random walk, *J. Analysis and its Applications (ZAA)* **18** No 2 (1999) 231–246.

[51] R. Gorenflo and F. Mainardi, Random walk models approximating symmetric space-fractional diffusion processes, in: J. Elschner, I. Gohberg and B. Silbermann (Editors), *Problems in Mathematical Physics* (Siegfried Prössdorf Memorial Volume), Birkhäuser Verlag, Boston-Basel-Berlin 2001, pp. 120-145

[52] R. Gorenflo and S. Vessella, *Abel Integral Equations: Analysis and Applications*, Springer-Verlag, Berlin 1991. [Lecture Notes in Mathematics, Vol. 1461]

[53] B. Gross, On creep and relaxation, *J. Appl. Phys.* **18** (1947) 212–221.

[54] A.K. Grünwald, Über "begrenzte" Derivation und deren Anwendung, *Z. angew. Math. und Phys.* **12** (1867) 441-480.

[55] K.G. Gupta and U.C. Jain, The H-function: II, *Proc. Nat. Acad. Sci. India* **A36** (1966) 594–602.

[56] R. Hilfer, Fractional Derivatives in Static and Dynamic Scaling, in B. Dubrulle, F. Graner and D. Sornette (Editors), *Scale Invariance and Beyond*, Springer

Verlag, Berlin and EDP Science, France, 1977, pp. 53-62. Lecture 3, Les Houches Workshop, March 10-14, 1997.

[57] R. Hilfer (Editor), *Applications of Fractional Calculus in Physics*, World Scientific, Singapore 2000.

[58] E. Hille and J.D. Tamarkin, On the theory of linear integral equations *Ann. Math.* **31** (1930) 479–528.

[59] Hj. Holmgren, Om differentialkalkylen med indices af hvad natur som helst, *Kongl. Svenska Vetenskaps-Akad. Hanl. Stockholm* **5** No.11 (1865-1866) 1–83.

[60] P. Humbert, Nouvelles correspondances symboliques, *Bull. Sci. Mathém.* (Paris, II ser.) **69** (1945) 121-129.

[61] P. Humbert, Quelques résultats relatifs à la fonction de Mittag-Leffler, *C.R. Acad. Sci. Paris* **236** (1953) 1467–1468.

[62] P. Humbert and R.P. Agarwal, Sur la fonction de Mittag-Leffler et quelques-unes de ses généralisations, *Bull. Sci. Math* (Ser. II) **77** (1953) 180–185.

[63] A.A. Kilbas, and M. Saigo, On Mittag-Leffler type functions, fractional calculus operators and solution of integral equations, *Integral Transforms and Special Functions* **4** (1996) 355-370.

[64] A.A. Kilbas and M. Saigo, On the H functions, *Journal of Applied Mathematics and Stochastic Analysis* **12** (1999) 191-204.

[65] A.A. Kilbas and M. Saigo, *H-transforms. Theory and Applications*, CRC Press, 2004.

[66] A.A. Kilbas, M. Saigo and J.J. Trujillo, On the generalized Wright function, *Fractional Calculus and Applied Analysis* **5** (2002) 437-460.

[67] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam 2006. [North-Holland Mathematics Studies No 204]

[68] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman, Harlow 1994. [Pitman Research Notes in Mathematics # 301]

[69] A.N. Kochubei, A Cauchy problem for evolution equations of fractional order, *Differential Equations* **25** (1989), 967–974. [English translation from the Russian Journal *Differentsial'nye Uravneniya*]

[70] A.N. Kochubei, Fractional order diffusion, *Differential Equations* **26** (1990), 485–492. [English translation from the Russian Journal *Differentsial'nye Uravneniya*

[71] A.V. Letnikov, Theory of differentiation with an arbitrary index, *Mat. Sb.* **3** (1868) 1–66. In Russian.

[72] A.V. Letnikov, On historical development of differentiation theory with an arbitrary index, *Mat. Sb.* **3** (1868) 85–112. [In Russian]

[73] J. Liouville, Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions, *J. l'École Roy. Polytéchn.* **13** Sect. 21 (1832) 1–69.

[74] J. Liouville, Mémoire sur le calcul des différentielles à indices quelconques, *J. l'École Roy. Polytéchn.* **13** Sect. 21 (1832) 71–162.

[75] J. Liouville, Mémoire sur l'intégration des équations différentielles à indices fractionnaires, *J. l'École Roy. Polytéchn.* **15** No 55 (1837) 55–84.

[76] Yu. Luchko, Asymptotics of zeros of the Wright function, *J. Analysis and its Applications (ZAA)* **19** (2000) 583–596.

[77] Yu. Luchko and R. Gorenflo, Scale-invariant solutions of a partial differential equation of fractional order, *Fractional Calculus and Applied Analysis* **1** (1998) 63-78.

[78] R.L. Magin, *Fractional Calculus in Bioengineering*, Begell Houuse Publishers, Connecticut, 2006.

[79] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, in: S. Rionero and T. Ruggeri (Editors), *Waves and Stability in Continuous Media*, World Scientific, Singapore 1994, pp. 246-251.

[80] F. Mainardi, Fractional diffusive waves in viscoelastic solids, in: J.L. Wegner and F.R. Norwood (Editors), *Nonlinear Waves in Solids*, ASME book No AMR 137, Fairfield NJ 1995, pp. 93–97. [Abstract in Appl. Mech. Rev. **46** No 12 (1993)) 549]

[81] F. Mainardi, The time fractional diffusion-wave equation, *Radiofisika* **38** No 1-2 (1995) 20–36. [English Transl.:*Radiophysics & Quantum Electronics*]

[82] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, *Applied Mathematics Letters* **9** No 6 (1996) 23–28.

[83] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons & Fractals* **7** (1996) 1461–1477.

[84] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in: A. Carpinteri and F. Mainardi (Editors), *Fractals and*

Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien 1997), pp. 291–248. [Reprinted in www.fracalmo.org]

[85] F. Mainardi and R. Gorenflo, The Mittag-Leffler function in the Riemann-Liouville fractional calculus, in: A.A. Kilbas (Editor), *Boundary Value Problems, Special Functions and Fractional Calculus*, Belarusian State University, Minsk 1996, pp. 215–225.

[86] F. Mainardi and R. Gorenflo, On Mittag-Leffler type functions in fractional evolution processes, *Journal of Computational and Applied Mathematics* **118** (2000) 283-299.

[87] F. Mainardi and R. Gorenflo, Fractional calculus: special functions and applications, in: G. Dattoli, H.M. Srivastava and D. Cocolicchio (Editors), *Advanced Special Functions and Applications*, Aracne, Roma 2000, pp. 165-189.

[88] F. Mainardi and G. Pagnini, : Salvatore Pincherle, the pioneer of the Mellin-Barnes integrals, *Journal of Computational and Applied Mathematics* **153** (2003) 331-342 .

[89] F. Mainardi and M. Tomirotti, On a special function arising in the time fractional diffusion-wave equation, in: P. Rusev, I. Dimovski and V. Kiryakova (Editors), *Transform Methods and Special Functions, Sofia 1994*, Science Culture Technology, Singapore 1995, pp. 171–183.

[90] F. Mainardi and M. Tomirotti, Seismic pulse propagation with constant Q and stable probability distributions, *Annali di Geofisica* **40** (1997) 1311-1328.

[91] A. Marchaud, Sur les dérivées et sur les différences des fonctions de vriables réelles, *J. Math. Pures et Appl.* **6** No 4 (1927) 337-425.

[92] A.M. Mathai and R.K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*, Wiley Eastern Ltd, New Delhi 1978.

[93] A.C. McBride, *Fractional Calculus and Integral Transforms of Generalized Functions*, Pitman, London 1979. [Pitman Research Notes in Mathematics # 31]

[94] A.C. McBride and G.F. Roach (Editors), *Fractional Calculus*, Proc. Int. Workshop. held at Univ. of Strathclyde (UK) in 1984, Pitmand, London 1985. [Pitman Research Notes in Mathematics # 138]

[95] J. Mikusiński, On the function whose Laplace transform is $\exp\left(-s^{\alpha}\lambda\right)$, *Studia Math.* **18** (1959) 191-198.

[96] K.S. Miller, The Weyl fractional calculus, in B. Ross (Editor), *Fractional Calculus and its Applications*, Springer Verlag, Berlin 1975, pp. 80-89. [Lecture Notes in Mathematics # 457]

[97] K.S. Miller and B. Ross: *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York 1993.

[98] G.M. Mittag-Leffler, Sur l'intégrale de Laplace-Abel, *C.R. Acad. Sci. Paris* (ser. II) **136** (1902) 937–939.

[99] G.M. Mittag-Leffler, Une généralisation de l'intégrale de Laplace-Abel, *C.R. Acad. Sci. Paris* (ser. II) **137** (1903) 537–539.

[100] G.M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C.R. Acad. Sci. Paris* (ser. II) **137** (1903) 554–558.

[101] G.M. Mittag-Leffler, Sopra la funzione $E_\alpha(x)$, *R. Accad. Lincei, Rend.* (ser. V) (1904) **13** 3–5.

[102] G.M. Mittag-Leffler, Sur la représentation analytique d'une branche uniforme d'une fonction monogène, *Acta Math.* **29** (1905) 101–181.

[103] K. Nishimoto (Editor), *Fractional Calculus and its Applications*, Nihon University, Tokyo 1990. [Proc. Int. Conf. held at Nihon Univ., Tokyo (Japan), 1989]

[104] K. Nishimoto, *An Essence of Nishimoto's Fractional Calculus*, Descartes Press, Koriyama 1991.

[105] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York 1974.

[106] S. Pincherle, Sulle funzioni ipergeometriche generalizzate, *Atti R. Accademia Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* (Ser. 4) **4** (1888) 694-700, 792-799. [Reprinted in *Salvatore Pincherle: Opere Scelte*, edited by UMI (Unione Matematica Italiana) Vol. 1, pp. 223-230, 231-239, Cremonese, Roma (1954)].

[107] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego 1999.

[108] H. Pollard, The representation of $\exp(-x^\lambda)$ as a Laplace integral, *Bull. Amer. Math. Soc.* **52** (1946) 908-910.

[109] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$, *Bull. Amer. Math. Soc.* **54** (1948) 1115–1116.

[110] B. Riemann, Versuch einer allgemeinen Auffassung der Integration und Differentiation, in *Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlikerNachlass* , pp 331–344, Teubner, Leipzig 1876. [New edition edited by H. Weber, Dover, New York 1953]

[111] M. Riesz, L'intégrales de Riemann-Liouville et le probléme de Cauchy, *Acta Math.* **81** No 1-2 (1949) 1-223.

[112] B. Ross (Editor), *Fractional Calculus and its Applications*, Springer Verlag, Berlin 1975. [Lecture Notes in Mathematics # 457]

[113] B. Rubin, *Fractional Integrals and Potentials*, Addison Wesley and Longman, Harlow 1996. [Pitman Monographs and Surveys in Pure and Applied Mathematics # 82]

[114] P. Rusev, I. Dimovski and V. Kiryakova (Editors), *Transform Methods and Special Functions, Sofia 1994*, Science Culture Technology, Singapore 1995. [Proc. 1-st Int. Workshop TMSF, Sofia, Bulgaria, 12-17 August 1994]

[115] P. Rusev, I. Dimovski, and V. Kiryakova (Editors), *Transform Methods and Special Functions, Varna 1996*, Inst. Maths & Informatics, Bulg. Acad. Sci, Sofia 1998). [Proc. 2-nd Int. Workshop TMSF, Varna, Bulgaria, 23-29 August 1996]

[116] A. Saichev and G. Zaslavsky, Fractional kinetic equations: solutions and applications, *Chaos* **7** (1997) 753-764.

[117] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Amsterdam 1993. [English translation from the Russian, *Integrals and Derivatives of Fractional Order and Some of Their Applications*, Nauka i Tekhnika, Minsk 1987.]

[118] G. Sansone and J. Gerretsen, *Lectures on the Theory of Functions of a Complex Variable*, Vol. I. *Holomorphic Functions*, Nordhoff, Groningen 1960, pp. 345–349.

[119] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge 1999.

[120] W.R. Schneider, Stable distributions: Fox function representation and generalization, in S. Albeverio, G. Casati and D. Merlini (Editors), *Stochastic Processes in Classical and Quantum Systems*, Springer Verlag, Berlin, pp. 497-511. [Lecture Notes in Physics, Vol. 262]

[121] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi, 1982.

[122] B. Stankoviċ, On the function of E.M. Wright, *Publ. de l'Institut Mathèmatique, Beograd, Nouvelle Sèr.* **10** (1970) 113-124.

[123] H. Takayasu, *Fractals in the Physical Sciences*, Manchester University Press, Manchester and New York 1990.

[124] F.G. Tricomi, *Fonctions Hypergéometriques Confluentes*, Mém. Sci. Math. # 140 (Gauthier-Villars, Paris, 1960).

[125] V.V. Uchaikin and V.M. Zolotarev, *Change and Stability. Stable Distributions and their Applications*, VSP, Utrecht 1999. [Series "Modern Probability and Statistics", No 3]

[126] Vu Kim Tuan and R. Gorenflo, Extrapolation to the limit for numerical fractional differentiation, *ZAMM* **75** (1995) 646-648.

[127] B.J. West, M. Bologna and P. Grigolini, *Physics of Fractal Operators*, Springer Verlag, New York 2003. [Institute for Nonlinear Science]

[128] U. Westphal, An approach to fractional powers of operators via fractional differences, *Proc. London Math. Soc.* **29** No 3 (1974) 557-576.

[129] U. Westphal, Fractional powers of infinitesimal generators of semigroups, in: R. Hilfer (Editor), *Applications of Fractional Calculus in Physics*, World Scientific, Singapore 2000, pp. 131-170.

[130] H. Weyl, Bemerkungen zum Begriff des Differentialquotientten gebrochener Ordnung, *Vierteljahresschrift der Naturforschenden Geselischaft in Zúrich* **62** No 1-2 (1917) 296-302.

[131] A. Wiman, Über die Nullstellen der Funktionen $E_\alpha[X]$, *Acta Math.* **29** (1905), 217–234.

[132] R. Wong and Y.-Q. Zhao, Smoothing of Stokes' discontinuity for the generalized Bessel function, *Proc. R. Soc. London A* **455** (1999) 1381–1400.

[133] R. Wong and Y.-Q. Zhao, Smoothing of Stokes' discontinuity for the generalized Bessel function II, *Proc. R. Soc. London A* **455** (1999) 3065–3084.

[134] E.M. Wright, On the coefficients of power series having exponential singularities, *Journal London Math. Soc.* **8** (1933)71-79.

[135] E.M. Wright, The asymptotic expansion of the generalized Bessel function, *Proc. London Math. Soc. (Ser. II)* **38** (1935) 257-270.

[136] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function, *Journal London Math. Soc.* **10** (1935) 287-293.

[137] E.M. Wright, The asymptotic expansion of the generalized Bessel function, *Proc. London Math. Soc. (Ser. II)* **46** (1940) 389-408.

[138] E.M. Wright, The generalized Bessel function of order greater than one, *Quart. J. Math., Oxford ser.* **11** (1940) 36-48.

[139] G.M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford 2005.

Additional references

[140] F. Mainardi, G. Pagnini and R.K. Saxena, Fox H functions in fractional diffusion, *J. Computational and Applied Mathematics* **178** No 1-2 (2005) 321-331.

[141] F. Mainardi, R. Gorenflo and A. Vivoli, Renewal processes of Mittag-Leffler and Wright type, *Fractional Calculus and Applied Analysis* **8** No 1 (2005) 7-38.. [E-print `http://arxiv.org/abs/math/0701455`]

[142] F. Mainardi, Applications of integral transforms in fractional diffusion processes, *Integral Transforms and Special Functions* **15** No 6 (2004) 477-484. [E-print `http://arxiv.org/abs//0710.0145`]

[143] F. Mainardi and G. Pagnini, The Wright functions as solutions of the time-fractional diffusion equations, *Applied Mathematics and Computation* **141** No 1 (2003) 51-66.

[144] [F. Mainardi and G. Pagnini, Salvatore Pincherle: the pioneer of the Mellin-Barnes integrals, *J. Computational and Applied Mathematics* **153** (2003) 331-342. [E-print `http://arxiv.org/abs/math/0702520`]

[145] F. Mainardi, G. Pagnini and R. Gorenflo, Mellin transform and subordination laws in fractional diffusion processes", *Fractional Calculus and Applied Analysis* **6** No. 4 (2003) 441-459. [E-print `http://arxiv.org/abs/math/0702133`]

[146] F. Mainardi, Yu. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fractional Calculus and Applied Analysis* **4** No 2 (2001) 153-192. [E-print `http://arxiv.org/abs/cond-mat/0702419`]