The functions of the Wright type in Fractional Calculus

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1. **Introduction**

Here we provide a survey of the high transcendental functions related to the Wright special function.

Like the functions of the Mittag-Leffler type, the functions of the Wright type are known to play fundamental roles in various applications of the fractional calculus. This is mainly due to the fact that they are interrelated with the Mittag-Leffler functions through a Laplace transformation.

We start providing the definitions in the complex plane for the general Wright function and for two special cases that we call auxiliary functions. Then we devote particular attention to the auxiliary functions in the real field, because they admit a probabilistic interpretation related to the fundamental solutions of certain evolution equations of fractional order. These equations are fundamental to understand phenomena of anomalous diffusion or intermediate between diffusion and wave propagation.

At the end we add some historical and bibliographical notes.
2. The Wright function $W_{\lambda, \mu}(z)$

The Wright function, that we denote by $W_{\lambda, \mu}(z)$ is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions, see [Wright (1933); 1935a; 1935b]. The function is defined by the series representation, convergent in the whole $z$-complex plane,

$$W_{\lambda, \mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}, \quad (F.1)$$

so $W_{\lambda, \mu}(z)$ is an entire function. Originally Wright assumed $\lambda > 0$, and, only in 1940, he considered $-1 < \lambda < 0$, see [Wright 1940]. We note that in the handbook of the Bateman Project [Erdelyi et al. Vol. 3, Ch. 18], presumably for a misprint, $\lambda$ is restricted to be non negative.
The integral representation. The integral representation reads

\[ W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu}, \quad \lambda > -1, \; \mu \in \mathbb{C}, \]  

where \( Ha \) denotes the Hankel path. The equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function

\[ \frac{1}{\Gamma(\zeta)} = \int_{Ha} e^u u^{-\zeta} du, \quad \zeta \in \mathbb{C}, \]

and performing a term-by-term integration. In fact,

\[
W_{\lambda,\mu}(z) = \frac{1}{2\pi i} \int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^\mu} = \frac{1}{2\pi i} \int_{Ha} e^{\sigma} \left[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{-\lambda n} \right] \frac{d\sigma}{\sigma^\mu}
\]

\[ = \sum_{n=0}^{\infty} \frac{z^n}{n!} \left[ \frac{1}{2\pi i} \int_{Ha} e^{\sigma} \sigma^{-\lambda n - \mu} d\sigma \right] = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma[\lambda n + \mu]}.
\]

It is possible to prove that the Wright function is entire of order \( 1/(1 + \lambda) \), hence of exponential type if \( \lambda \geq 0 \). The case \( \lambda = 0 \) is trivial since \( W_{0,\mu}(z) = e^z / \Gamma(\mu) \).
Asymptotic expansions. For the detailed asymptotic analysis in the whole complex plane for the Wright functions, the interested reader is referred to [Wong and Zhao (1999a); (1999b)]. These Authors have considered the two cases $\lambda \geq 0$ and $-1 < \lambda < 0$ separately, including a description of Stokes’ discontinuity and its smoothing.

In the second case, that, as a matter of fact is the most interesting for us, we set $\lambda = -\nu \in (-1, 0)$, and we recall the asymptotic expansion originally obtained by Wright himself, that is valid in a suitable sector about the negative real axis as $|z| \to -\infty$,

$$W_{-\nu, \mu}(z) = Y^{1/2-\mu} \exp \left[-Y \sum_{m=0}^{M-1} A_m Y^{-m} + O(|Y|^{-M}) \right],$$

$$Y = Y(z) = (1 - \nu) (-\nu^\nu z)^{1/(1-\nu)}, \quad (F.3)$$

where the $A_m$ are certain real numbers.
Generalization of the Bessel functions. For $\lambda = 1$ and $\mu = \nu + 1 \geq 0$ the Wright functions turn out to be related to the well known Bessel functions $J_\nu$ and $I_\nu$ by the identities:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(-\frac{z^2}{4}\right), \quad I_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(\frac{z^2}{4}\right).$$

(F.4)

In view of this property some authors refer to the Wright function as the *Wright generalized Bessel function* (misnamed also as the *Bessel-Maitland function*) and introduce the notation

$$J_\nu^{(\lambda)}(z) := \left(\frac{z}{2}\right)^\nu W_{\lambda,\nu+1}\left(-\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(\lambda n + \nu + 1)},$$

(F.5)

with $\lambda > 0$ and $\nu > -1$. In particular $J_\nu^{(1)}(z) := J_\nu(z)$. As a matter of fact, the Wright function appears as the natural generalization of the entire function known as *Bessel - Clifford function*, see e.g. [Kiryakova 1999, p. 336], and referred by Tricomi as the *uniform Bessel function*. Similarly we can properly define $I_\nu^{(\lambda)}(z)$.
Some of the properties, that the Wright functions share with the most popular Bessel functions, were enumerated by Wright himself.

Hereafter, we quote some relevant relations from the handbook of Bateman Project Handbook, see [Erdelyi 1955, Vol. 3, Ch. 18]:

\[ \lambda z W_{\lambda,\lambda+\mu}(z) = W_{\lambda,\mu-1}(z) + (1 - \mu) W_{\lambda,\mu}(z) , \quad (F.7) \]

\[ \frac{d}{dz} W_{\lambda,\mu}(z) = W_{\lambda,\lambda+\mu}(z) . \quad (F.8) \]

We note that these relations can easily be derived from the series or integral representations, (F.1) or (F.2).
3. The Auxiliary Functions $F_\nu(z)$ and $M_\nu(z)$ in the complex plane

In his first analysis of the time fractional diffusion equation the Author [Mainardi 1993-1994], aware of the Bateman project but not of 1940 paper by Wright, introduced the two (Wright-type) entire auxiliary functions,

$$F_\nu(z) := W_{-\nu,0}(-z), \ 0 < \nu < 1,$$  \hspace{0.5cm} (F.9)

and

$$M_\nu(z) := W_{-\nu,1-\nu}(-z), \ 0 < \nu < 1,$$ \hspace{0.5cm} (F.10)

inter-related through

$$F_\nu(z) = \nu z M_\nu(z).$$  \hspace{0.5cm} (F.11)

As a matter of fact the functions $F_\nu(z)$ and $M_\nu$ are particular cases of $W_{\lambda,\mu}(z)$ by setting $\lambda = -\nu$ and $\mu = 0, \mu = 1$, respectively.
Hereafter we provide the series and integral representations of the two auxiliary functions derived from the general formulas (F.1) and (F.2), respectively.

**Series representations.** The *series representations* for our auxiliary functions read

\[
F_\nu(z) := \sum_{n=1}^{\infty} \frac{(-z)^n}{n! \Gamma(-\nu n)},
\]

\[
::= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{n!} \Gamma(\nu n + 1) \sin(\pi \nu n),
\]

and

\[
M_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma[-\nu n + (1 - \nu)]},
\]

\[
::= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n),
\]
The second series representations in (F.11)-(F.12) have been obtained by using the well-known reflection formula for the Gamma function,

$$\Gamma(\zeta) \Gamma(1 - \zeta) = \frac{\pi}{\sin \pi \zeta}.$$  

As an exercise the reader can easily prove that the radius of convergence of the power series in (F.12)-(F.13) if infinite for $0 < \nu < 1$, without be aware of the Wright functions, see also [Podlubny (1999)].

Furthermore we note that $F_\nu(0) = 0$, $M_\nu(0) = 1/\Gamma(1 - \nu)$

As far as the the relation (F.11) is concerned, it can be easily deduced from (F.12)-(F.13) by using the basic property of the Gamma function or recalling the relation (F.7) setting $\lambda = -\nu$ and $\mu = 1$. 
The integral representations. The *integral representations* for our auxiliary functions read:

\[
F_\nu(z) := \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\nu} d\sigma, \quad (F.14)
\]

\[
M_\nu(z) := \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}}, \quad (F.15)
\]

We note that the relation (F.11), \( F_\nu(z) = \nu z M_\nu(z) \), can be obtained directly from (F.12)-(F.13) with an integration by parts. In fact,

\[
M_\nu(z) = \int_{Ha} e^{\sigma - z\sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}} = \int_{Ha} e^{\sigma} \left( -\frac{1}{\nu z} \frac{d}{d\sigma} e^{-z\sigma^\nu} \right) d\sigma
\]

\[
= \frac{1}{\nu z} \int_{Ha} e^{\sigma - z\sigma^\nu} d\sigma = \frac{F_\nu(z)}{\nu z}.
\]

As usual, the equivalence of the series and integral representations is easily proved using the Hankel formula for the Gamma function and performing a term-by-term integration.
Special cases. Explicit expressions of $F_\nu(z)$ and $M_\nu(z)$ in terms of known functions are expected for some particular values of $\nu$. In [Mainardi & Tomirotti 1994] the Authors have shown that for $\nu = 1/q$, where $q \geq 2$ is a positive integer, the auxiliary functions can be expressed as a sum of $(q-1)$ simpler entire functions. In the particular cases $q = 2$ and $q = 3$ we find

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{2}\right)_m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}} \exp \left(-\frac{z^2}{4}\right),$$

and

$$M_{1/3}(z) = \frac{1}{\Gamma(2/3)} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)_m \frac{z^{3m}}{(3m)!} - \frac{1}{\Gamma(1/3)} \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)_m \frac{z^{3m+1}}{(3m+1)!}$$

$$= 3^{2/3} \text{Ai}\left(\frac{z}{3^{1/3}}\right),$$

where $\text{Ai}$ denotes the Airy function.
Furthermore, it can be proved that $M_{1/q}(z)$ satisfies the differential equation of order $q - 1$

$$
\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q} z M_{1/q}(z) = 0,
$$

subjected to the $q - 1$ initial conditions at $z = 0$, derived from (F.15),

$$
M_{1/q}^{(h)}(0) = \frac{(-1)^h}{\pi} \Gamma[(h + 1)/q] \sin[\pi (h + 1)/q],
$$

with $h = 0, 1, \ldots q - 2$. We note that, for $q \geq 4$, Eq. (F.18) is akin to the hyper-Airy differential equation of order $q - 1$, see e.g. [Bender & Orszag 1987]. Consequently, in view of the above considerations, the auxiliary function $M_\nu(z)$ can be referred to as the generalized hyper-Airy function.
4. The Auxiliary Functions $F_\nu(x)$ and $M_\nu(x)$ on the real axis

We point out that the most relevant applications of Wright functions, specially our auxiliary functions, are when the variable is real.

More precisely, from now on, we will consider functions that are defined either on the positive real semi-axis $\mathbb{R}^+$ or on all of $\mathbb{R}$ in a symmetric way.

We agree to denote the variable with $r$ when is restricted to $\mathbb{R}^+$ and with $x$ for $\mathbb{R}$. Of course, sometimes $x$ may be replaced by $|x|$ in order to stress the symmetry of the function in $\mathbb{R}$. 
The asymptotic representation of $M_\nu(r)$. Let us first point out the asymptotic behaviour of the function $M_\nu(r)$ as $r \to \infty$. Choosing as a variable $r/\nu$ rather than $r$, the computation of the requested asymptotic representation by the addle-point approximation yields, see [Mainardi & Tomirotti 1994],

$$M_\nu(r/\nu) \sim a(\nu) r^{(\nu - 1/2)/(1 - \nu)} \exp \left[ -b(\nu) r^{1/(1 - \nu)} \right],$$

where

$$a(\nu) = \frac{1}{\sqrt{2\pi (1 - \nu)}} > 0, \quad b(\nu) = \frac{1 - \nu}{\nu} > 0. \quad (F.20)$$

The above evaluation is consistent with the first term in Wright’s asymptotic expansion (F.3) after having used the definition (F.10).

We point out that in the limit $\nu \to 1^-$ the function $M_\nu(r)$ tends to the Dirac generalized function $\delta(r - 1)$. 
Plots of $M_\nu(x)$. We find it instructive to show the plots of our auxiliary functions on the real axis for some rational values of the parameter $\nu$. To gain more insight of the effect of the parameter itself on the behaviour close to and far from the origin, we will adopt both linear and logarithmic scale for the ordinates.

In Figs. F.1 and F.2 we compare the plots of the $M_\nu(x)$-Wright functions in $-5 \leq x \leq 5$ for some rational values in the ranges $\nu \in [0, 1/\nu]$ and $\nu \in [1/2, 1]$, respectively. Thus in Fig. F.1 we see the transition from $\exp(-|x|)$ for $\nu = 0$ to $1/\sqrt{\pi} \exp(-x^2)$ for $\nu = 1/2$, whereas in Fig. F.2 we see the transition from $1/\sqrt{\pi} \exp(-x^2)$ to the delta functions $\delta(x \pm 1)$ for $\nu = 1$.

In plotting $M_\nu(x)$ at fixed $\nu$ for sufficiently large $x$ the asymptotic representation (F.20)-(F.21) is useful since, as $x$ increases, the numerical convergence of the series in (F.15) becomes poor and poor up to being completely inefficient: henceforth, the matching between the series and the asymptotic representation is relevant. However, as $\nu \to 1^-$, the plotting remains a very difficult task because of the high peak arising around $x = \pm 1$. 
Figure 1: Plots of the Wright type function $M_{\nu}(x)$ with $\nu = 0, 1/8, 1/4, 3/8, 1/2$ for $-5 \leq x \leq 5$; top: linear scale, bottom: logarithmic scale.
Figure 2: Plots of the Wright type function $M_\nu(x)$ with $\nu = 1/2, 5/8, 3/4, 1$ for $-5 \leq x \leq 5$: top: linear scale; bottom: logarithmic scale).
5. The Laplace transform pairs

Let us consider the Laplace transform of the Wright function using the following notation

\[ W_{\lambda,\mu}(\pm r) \div \mathcal{L} \left[ W_{\lambda,\mu}(\pm r); s \right] := \int_0^\infty e^{-s r} W_{\lambda,\mu}(\pm r) \, dr , \]

where \( r \) denotes a non negative real variable, \( i.e. \) \( 0 \leq r < +\infty \), and \( s \) is the Laplace complex parameter.

When \( \lambda > 0 \) the series representation of the Wright function can be transformed term-by-term. In fact, for a known theorem of the theory of the Laplace transforms, see \( e.g. \) [Doetsch (197], the Laplace transform of an entire function of exponential type can be obtained by transforming term-by-term the Taylor expansion of the original function around the origin. In this case the resulting Laplace transform turns out to be analytic and vanishing at infinity.
As a consequence, we obtain the Laplace transform pair

\[
W_{\lambda,\mu}(\pm r) \div \frac{1}{s} E_{\lambda,\mu}\left(\pm \frac{1}{s}\right), \quad \lambda > 0, \quad |s| > 0,
\]

(F.22)

where \(E_{\lambda,\mu}\) denotes the Mittag-Leffler function in two parameters. The proof is straightforward noting that

\[
\sum_{n=0}^{\infty} \frac{(\pm r)^n}{n! \Gamma(\lambda n + \mu)} \div \frac{1}{s} \sum_{n=0}^{\infty} \frac{(\pm 1/s)^n}{\Gamma(\lambda n + \mu)},
\]

and recalling the series representation of the Mittag-Leffler function,

\[
E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{C}.
\]
For $\lambda \to 0^+$ Eq. (F.22) provides the Laplace transform pair
\[
W_{0+},\mu(\pm r) = \frac{e^{\pm r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s \mp 1} = \frac{1}{s} E_{0,\mu} \left( \pm \frac{1}{s} \right), \quad |s| > 1,
\]
where, to remain in agreement with (F.22), we have formally put
\[
E_{0,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu)} := \frac{1}{\Gamma(\mu)} E_0(z) := \frac{1}{\Gamma(\mu)} \frac{1}{1 - z}, \quad |z| < 1.
\]
We recognize that in this special case the Laplace transform exhibits a simple pole at $s = \pm 1$ while for $\lambda > 0$ it exhibits an essential singularity at $s = 0$. 
For $-1 < \lambda < 0$ the Wright function turns out to be an entire function of order greater than 1, so that care is required in establishing the existence of its Laplace transform, which necessarily must tend to zero as $s \to \infty$ in its half-plane of convergence.

For the sake of convenience we limit ourselves to derive the Laplace transform for the special case of $M_\nu(r)$; the exponential decay as $r \to \infty$ of the original function provided by (F.20) ensures the existence of the image function. From the integral representation (F.13) of the $M_\nu$ function we obtain

$$M_\nu(r) \div \frac{1}{2\pi i} \int_0^\infty e^{-s \cdot r} \left[ \int_{H_a} e^{\sigma - r \sigma^\nu} \frac{d\sigma}{\sigma^{1-\nu}} \right] dr$$

$$= \frac{1}{2\pi i} \int_{H_a} e^{\sigma \sigma^\nu-1} \left[ \int_0^\infty e^{-r(s + \sigma^\nu)} dr \right] d\sigma = \frac{1}{2\pi i} \int_{H_a} \frac{e^{\sigma \sigma^\nu-1}}{\sigma^\nu + s} d\sigma.$$
Then, by recalling the integral representation of the Mittag-Leffler function,

\[ E_\alpha(z) = \frac{1}{2\pi i} \int_{H_a} \frac{\zeta^{\alpha-1} e^\zeta}{\zeta^\alpha - z} \, d\zeta, \quad \alpha > 0, \ z \in \mathbb{C}, \]

we obtain the Laplace transform pair

\[ M_\nu(r) := W_{-\nu,1-\nu}(-r) \div E_\nu(-s), \ 0 < \nu < 1. \quad (F.24) \]

In this case, transforming term-by-term the Taylor series of \( M_\nu(r) \) yields a series of negative powers of \( s \), that represents the asymptotic expansion of \( E_\nu(-s) \) as \( s \to \infty \) in a sector around the positive real axis.

We note that (F.24) contains the well-known Laplace transform pair, see e.g. [Doetsch 1974],

\[ M_{1/2}(r) := \frac{1}{\sqrt{\pi}} \exp \left(-\frac{r^2}{4}\right) \div E_{1/2}(-s) := \exp \left(s^2\right) \text{erfc} \left(s\right), \]

valid \( \forall s \in \mathbb{C} \).
Analogously, using the more general integral representation (F.2) of the standard Wright function, we can prove that in the case $\lambda = -\nu \in (-1, 0)$ and $\mathcal{R}\mu > 0$, we get

$$W_{-\nu,\mu}(-r) \div E_{\nu,\mu+\nu}(-s), \ 0 < \nu < 1. \quad (F.25)$$

In the limit as $\nu \to 0^+$ (thus $\lambda \to 0^-$) we formally obtain the Laplace transform pair

$$W_{0-,\mu}(-r) := \frac{e^{-r}}{\Gamma(\mu)} \div \frac{1}{\Gamma(\mu)} \frac{1}{s + 1} := E_{0,\mu}(-s). \quad (F.26)$$

Therefore, as $\lambda \to 0^\pm$, and $\mu = 1$ we note a sort of continuity in the results (F.23) and (F.26) since

$$W_{0,1}(-r) := e^{-r} \div \frac{1}{(s + 1)} = \begin{cases} (1/s) \ E_0(-1/s), \ |s| > 1; \\ E_0(-s), \ |s| < 1. \end{cases} \quad (F.27)$$
We here point out the relevant Laplace transform pairs related to the auxiliary functions of argument $r^{-\nu}$, see [Mainardi (1994); (1996a); (1996b)],

$$\frac{1}{r} F_{\nu} \left( 1/r^{\nu} \right) = \frac{\nu}{r^{\nu+1}} M_{\nu} \left( 1/r^{\nu} \right) \div e^{-s^{\nu}}, \ 0 < \nu < 1. \quad (F.28)$$

$$\frac{1}{\nu} F_{\nu} \left( 1/r^{\nu} \right) = \frac{1}{r^{\nu}} M_{\nu} \left( 1/r^{\nu} \right) \div \frac{e^{-s^{\nu}}}{s^{1-\nu}}, \ 0 < \nu < 1. \quad (F.29)$$

We recall that the Laplace transform pairs in (F.28) were formerly considered by [Pollard (1946)], who provided a rigorous proof based on a formal result by [Humbert (1945)]. Later [Mikusinski (1959)] got a similar result based on his theory of operational calculus, and finally, albeit unaware of the previous results, [Buchen & Mainardi (1975)] derived the result in a formal way. We note, however, that all these Authors were not informed about the Wright functions.
Hereafter we like to provide two independent proofs of (F.28) carrying out the inversion of \( \exp (-s^\nu) \), either by the complex Bromwich integral formula or by the formal series method. Similarly we can act for the Laplace transform pair (F.29).

For the complex integral approach we deform the Bromwich path \( Br \) into the Hankel path \( Ha \), that is equivalent to the original path, and we set \( \sigma = sr \). Recalling (F.13)-(F.14), we get

\[
\mathcal{L}^{-1} \left[ \exp (-s^\nu) \right] = \frac{1}{2\pi i} \int_{Br} e^{sr - s^\nu} ds = \frac{1}{2\pi i} \int_{Ha} e^{\sigma} - (\sigma/r)^\nu d\sigma
\]

\[
= \frac{1}{r} F_\nu \left( 1/r^\nu \right) = \frac{\nu}{r^{\nu+1}} M_\nu \left( 1/r^\nu \right).
\]

Expanding in power series the Laplace transform and inverting term by term we formally get, after recalling (F.12)-(F.13):

\[
\mathcal{L}^{-1} \left[ \exp (-s^\nu) \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{L}^{-1} \left[ s^{\nu n} \right] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{r^{-\nu n-1}}{\Gamma(-\nu n)}
\]

\[
= \frac{1}{r} F_\nu \left( 1/r^\nu \right) = \frac{\nu}{r^{\nu+1}} M_\nu \left( 1/r^\nu \right).
\]
We note the relevance of Laplace transforms (F.24) and (F.28) in pointing out the non-negativity of the Wright function $M_\nu(x)$ and the complete monotonicity of the Mittag-leffler functions $E_\nu(-x)$ for $x > 0$ and $0 < \nu < 1$. In fact, since $\exp(-s^\nu)$ denotes the Laplace transform of a probability density (precisely, the extremal Lévy stable density of index $\nu$, see [Feller (1971)]), the L.H.S. of (F.28) must be non-negative, and so also the L.H.S of F(24). As a matter of fact the Laplace transform pair (F.24) shows, replacing $s$ by $x$, that the spectral representation of the Mittag-Leffler function $E_\nu(-x)$ is expressed in terms of the $M$-Wright function $M_\nu(r)$, that is:

$$E_\nu(-x) = \int_0^\infty e^{-r x} M_\nu(r) \, dr, \quad 0 < \nu < 1, \quad x \geq 0. \quad (F.30)$$

We now recognize that Eq. (F.30) is consistent with a result derived by [Pollard (1948)].
It is instructive to compare the spectral representation of $E_{\nu}(-x)$ with that of the function $E_{\nu}(-t^\nu)$. We recall

$$E_{\nu}(-t^\nu) = \int_0^\infty e^{-rt} K_{\nu}(r) \, dr, \quad 0 < \nu < 1, \, t \geq 0, \quad (F.31)$$

with spectral function

$$K_{\nu}(r) = \frac{1}{\pi} \frac{r^{\nu-1} \sin(\nu\pi)}{r^{2\nu} + 2r^\nu \cos(\nu\pi) + 1} = \frac{1}{\pi r} \frac{\sin(\nu\pi)}{r^\nu + r^{-\nu} + 2 \cos(\nu\pi)}. \quad (F.32)$$

The relationship between $M_{\nu}(r)$ and $K_{\nu}(r)$ is worth to be explored. Both functions are non-negative, integrable and normalized in $\mathbb{IR}^+$, so they can be adopted in probability theory as density functions.

We have already shown the plot of $M_{\nu}(r)$, see Figs. 1, 2 with $r = |x|$. The plot of $K_{\nu}(r)$ is the same of $R_*(\tau)$ of the spectrum in the fractional Zener model in linear viscoelasticitymm, see Fig. 3.
Figure 3: Plots of the spectral function $R_\ast(\tau)$ in $0 \leq \tau \leq 2$ for $\nu = 0.25, 0.50, 0.75, 0.90$
Whereas the transition $K_{\nu}(r) \to \delta(r - 1)$ for $\nu \to 1$ is easy to be detected numerically in view of the explicit representation (F.32), the analogous transition $M_{\nu}(r) \to \delta(r - 1)$ is quite a difficult matter in view of its series and integral representations. In this respect see Fig. 4 from [Mainardi-Tomirotti (1997)]:
Figure 4: Comparison of the representations of $M_\nu(r)$ with $\nu = 1 - \epsilon$ around the maximum $r \approx 1$, in the cases (a) $\epsilon = 0.01$, (b) $\epsilon = 0.001$, obtained by Pipkin’s method (continuous line), 100 terms-series (dashed line) and and saddle-point method (dashed-dotted line).
6. The $M_\nu$-Wright functions in Probability

The $M_\nu$-Wright functions play fundamental roles in Theory of Probability and Stochastic Processes with support both in $\mathbb{R}^+$ (the variable is a time coordinate) and in all of $\mathbb{R}$ (the variable is the absolute value of a space coordinate).

For certain stochastic processes of renewal type, functions of Mittag-Leffler and Wright type can be adopted as probability distributions, see [Mainardi-Gorenflo-Vivoli (FCAA 2005)], where they are compared.

Here, from now on, we agree to denote by $x$ the variable of $M_\nu$ functions even when restricted in $\mathbb{R}^+$. The exponential decay for $x \to +\infty$ pointed out in Eqs (F.20)-(F.21) ensures that $M_\nu(x)$ is absolutely integrable in $\mathbb{R}^+$ and in $\mathbb{R}$. By recalling the Laplace transform pair (F.24) related to the Mittag-Leffler function, we get

$$\int_0^{+\infty} M_\nu(x) \, dx = \frac{1}{2} \int_{-\infty}^{+\infty} M_\nu(|x|) \, dx = E_\nu(0) = 1.$$  \hfill (F.33)
Being non-negative, $M_\nu(x)$ and $\frac{1}{2}M_\nu(|x|)$ can be interpreted as probability density functions in $\mathbb{R}^+$ and in $\mathbb{R}$, respectively. More generally, we can compute from (F.24) all the moments in $\mathbb{R}^+$, for $n = 1, 2 \ldots$, as follows

$$\int_0^{+\infty} x^n M_\nu(x) \, dx = \lim_{s \to 0} (-1)^n \frac{d^n}{ds^n} E_\nu(-s) = \frac{\Gamma(n + 1)}{\Gamma(\nu n + 1)}.$$

(E.34)

An alternative proof of Eqs. (F.33)-(F.34) can be sketched as follows:

$$\int_0^{+\infty} x^n M_\nu(x) \, dx = \frac{1}{2\pi i} \int_{Ha} e^\sigma \left[ \int_0^{+\infty} e^{-\sigma x} x^n \, dx \right] \frac{d\sigma}{\sigma^{1-\nu}}$$

$$= \frac{n!}{2\pi i} \int_{Ha} \frac{e^\sigma}{\sigma^{\nu n + 1}} \, d\sigma = \frac{\Gamma(n + 1)}{\Gamma(\nu n + 1)},$$

where the exchange between the two integrals turns out to be legitimate.
The characteristic function of the symmetric $M$-Wright function. The Fourier transform of the symmetric (and normalized) $M_\nu$-Wright function provides its characteristic function useful in Probability theory.

$$
\mathcal{F} \left[ \frac{1}{2} M_\nu(|x|) \right] := \frac{1}{2} \int_{-\infty}^{+\infty} e^{i\kappa x} M_\nu(|x|) \, dx \\
= \int_{0}^{\infty} \cos(\kappa x) M_\nu(x) \, dx = E_{2\nu}(-\kappa^2).
$$

For this prove it is sufficient to develop in series the cosine function and use the formula for the absolute moments of the $M$-Wright function in $\mathbb{R}^+$. 

$$
\int_{0}^{\infty} \cos(\kappa x) M_\nu(x) \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n}}{(2n)!} \int_{0}^{\infty} x^{2n} M_\nu(x) \, dx \\
= \sum_{n=0}^{\infty} (-1)^n \frac{\kappa^{2n}}{\Gamma(2\nu n + 1)} = E_{2\nu}(-\kappa^2).
$$
Relations with Lévy stable distributions. We find it worth to discuss the relations between the $M_\nu$-Wright functions and the so called Lévy stable distributions.

The term stable has been assigned by the French mathematician Paul Lévy, who in the 1920’s years started a systematic research in order to generalize the celebrated Central Limit Theorem to probability distributions with infinite variance. For stable distributions we can assume the following Definition:

*If two independent real random variables with the same shape or type of distribution are combined linearly and the distribution of the resulting random variable has the same shape, the common distribution (or its type, more precisely) is said to be stable.*

The restrictive condition of stability enabled Lévy (and then other authors) to derive the *canonic form* for the Fourier transform of the densities of these distributions. Such transform in probability theory is known as *characteristic function*. 
Here we follow the parameterization in [Feller (1952); (1971)] revisited in [Gorenflo & Mainardi (FCAA 1998)] and in [Mainardi-Luchko-Pagnini (FCAA 2001)].

Denoting by $L_{\theta}^{\alpha}(x)$ a generic stable density in IR, where $\alpha$ is the index of stability and and $\theta$ the asymmetry parameter, improperly called skewness, its characteristic function reads:

$$L_{\alpha}^{\theta}(x) = \frac{\hat{L}_{\alpha}^{\theta}(\kappa)}{\hat{L}_{\alpha}^{\theta}(\kappa)} = \exp \left[ -\psi_{\alpha}^{\theta}(\kappa) \right], \quad \psi_{\alpha}^{\theta}(\kappa) = |\kappa|^\alpha e^{i(\text{sign } \kappa)\theta \pi/2},$$

$$(F.35) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 < \alpha \leq 2, \; |\theta| \leq \min \{\alpha, 2 - \alpha\}.$$ 

We note that the allowed region for the parameters $\alpha$ and $\theta$ turns out to be a diamond in the plane $\{\alpha, \theta\}$ with vertices in the points $(0, 0), (1, 1), (1, -1), (2, 0)$, that we call the Feller-Takayasu diamond, see Fig. F.4. For values of $\theta$ on the border of the diamond (that is $\theta = \pm \alpha$ if $0 < \alpha < 1$, and $\theta = \pm (2 - \alpha)$ if $1 < \alpha < 2$) we obtain the so-called extremal stable densities.

We note the symmetry relation $L_{\alpha}^{\theta}(-x) = L_{\alpha}^{-\theta}(x)$, so that a stable density with $\theta = 0$ is symmetric
Figure 5: The Feller-Takayasu diamond for Lévy stable densities.
Stable distributions have noteworthy properties of which the interested reader can be informed from the existing literature. Hereafter we recall some peculiar PROPERTIES:

- The class of stable distributions possesses its own domain of attraction, see e.g. [Feller (1971)].
- Any stable density is unimodal and indeed bell-shaped, i.e. its $n$-th derivative has exactly $n$ zeros in $\mathbb{R}$, see [Gawronski (1984)].
- The stable distributions are self-similar and infinitely divisible.

These properties derive from the canonic form (F.35) through the scaling property of the Fourier transform. Self-similarity means

$$L_\alpha(x, t) \div \exp \left[ -t\psi_\alpha'(\kappa) \right] \iff L_\alpha(x, t) = t^{-1/\alpha} L_\alpha(x/t^{1/\alpha}),$$

(F.36)

where $t$ is a positive parameter. If $t$ is time, then $L_\alpha^\theta(x, t)$ is a spatial density evolving on time with self-similarity.
Infinite divisibility means that for every positive integer $n$, the characteristic function can be expressed as the $n$th power of some characteristic function, so that any stable distribution can be expressed as the $n$-fold convolution of a stable distribution of the same type. Indeed, taking in (F.35) $\theta = 0$, without loss of generality, we have

$$e^{-t|\kappa|^{\alpha}} = \left[e^{-\left(t/n\right)|\kappa|^{\alpha}}\right]^n \iff L_\alpha(x, t) = \left[L_\alpha(x, t/n)\right]^n, \ (F.37)$$

where

$$\left[L_\alpha(x, t/n)\right]^n := L_\alpha(x, t/n) \ast L_\alpha(x, t/n) \ast \cdots \ast L_\alpha(x, t)$$

is the multiple Fourier convolution in $\mathbb{R}$ with $n$ identical terms.
Only for a few particular cases, the inversion of the Fourier transform in (F.35) can be carried out using standard tables, and well-known probability distributions are obtained.

For $\alpha = 2$ (so $\theta = 0$), we recover the *Gaussian density*, that turns out to be the only stable density with finite variance, and more generally with finite moments of any order $\delta \geq 0$.

All the other stable densities have finite absolute moments of order $\delta \in [0, \alpha)$.

For $\alpha = 1$ we get: if $\theta = 0$, the *Cauchy-Lorentz density*, whereas, if $\theta = \pm 1$, the singular densities $\delta(x \pm 1)$.

In general we must recall the power series expansions provided in [Feller (1971)]. We restrict our attention to $x > 0$ since the evaluations for $x < 0$ can be obtained using the symmetry relation.
The convergent expansions of $L_\theta^\alpha(x)$ ($x > 0$) turn out to be for $0 < \alpha < 1$, $|\theta| \leq \alpha$:

$$L_\theta^\alpha(x) = \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \frac{\Gamma(1+n\alpha)}{n!} \sin \left(\frac{n\pi}{2} (\theta - \alpha)\right);$$

(F.38)

for $1 < \alpha \leq 2$, $|\theta| \leq 2 - \alpha$:

$$L_\theta^\alpha(x) = \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x)^n \frac{\Gamma(1+n/\alpha)}{n!} \sin \left(\frac{n\pi}{2\alpha} (\theta - \alpha)\right).$$

(F.39)

From the series in (F.38) and the symmetry relation we note that the extremal stable densities for $0 < \alpha < 1$ are unilateral, precisely vanishing for $x > 0$ if $\theta = \alpha$, vanishing for $x < 0$ if $\theta = -\alpha$. In particular the unilateral extremal densities $L_{-\alpha}^{-\alpha}(x)$ with $0 < \alpha < 1$ have as Laplace transform $\exp(-s^{\alpha})$. 
From a comparison between the series expansions in (F.38)-(F.39) and in (F.14)-(F.15), we recognize that for $x > 0$ our auxiliary functions of the Wright type are related to the extremal stable densities as follows, see [Mainardi-Tomirotti (1997)],

$$L^{-\alpha}(x) = \frac{1}{x} \frac{\alpha}{x^{\alpha+1}} M_{\alpha}(x^{-\alpha}), \quad 0 < \alpha < 1, \quad (F.40)$$

$$L^{\alpha-2}(x) = \frac{1}{x} \frac{1}{\alpha} M_{1/\alpha}(x), \quad 1 < \alpha \leq 2. \quad (F.41)$$

In Eqs. (F.40)-(F.41), for $\alpha = 1$, the skewness parameter turns out to be $\theta = -1$, so we get the singular limit $L^{-1}(x) = M_1(x) = \delta(x - 1)$.

More generally, all (regular) stable densities, given in Eqs. (F.38)-(F.39), were recognized to belong to the class of Fox $H$-functions, as formerly shown by [Schneider (LNP 1986)], see also [Mainardi-Pagnini-Saxena (2003)]. This general class of high transcendental functions is out of the scope of this book.
The subordination formula for the $M_\nu$-Wright functions. We now consider $M$-Wright functions as spatial probability densities evolving in time with self-similarity, that is

$$M_\nu(x, t) := t^{-\nu} M_\nu(x t^{-\nu}), \quad x, t \geq 0.$$  \hspace{1cm} (F.42)

These $M$-Wright functions are relevant for their composition rules proved in [Mainardi-Luchko-Pagnini (FCAA 2001)] and more generally in [Mainardi-Pagnini-Gorenflo (FCAA 2003)] by using the Mellin Transforms.

The main statement can be summarized with the THEOREM: Let $M_\lambda(x; t)$, $M_\mu(x; t)$ and $M_\nu(x; t)$ be $M$-Wright functions of orders $\lambda, \mu, \nu \in (0, 1)$ respectively, then the following composition formula holds true for any $x, t \geq 0$:

$$M_\nu(x, t) = \int_0^\infty M_\lambda(x; \tau) M_\mu(\tau; t) \, d\tau, \quad \text{with} \quad \nu = \lambda \mu.$$  \hspace{1cm} (F.43)
The above equation is also intended as a subordination formula because it can be used to define subordination among self-similar stochastic processes (with independent increments), that properly generalize the most popular Gaussian processes, to which they reduce for $\nu = 1/2$.

These more general processes are governed by time-fractional diffusion equations, as shown in recent papers of our research group, see [Mura-Pagnini (JPhysA 2008)], [Mura-Taqqu-Mainardi (PhysicaA 2008)], to which the interested reader is referred for details.
7. Notes

In early nineties, in his former analysis of fractional equations interpolating diffusion and wave-propagation, the present Author, see e.g. [Mainardi (WASCOM 1993)], has introduced the functions of the Wright type

$$F_{\nu}(z) := \Phi_{-\nu,0}(-z), \quad M_{\nu}(z) := \Phi_{-\nu,1-\nu}(-z)$$

with $0 < \nu < 1$, inter-related through $F_{\nu}(z) = \nu z M_{\nu}(z)$ to characterize the solutions for typical boundary value problems.

Being in that time only aware of the Bateman project where the parameter $\lambda$ of the Wright function $\Phi_{\lambda,\mu}(z)$ was erroneously restricted to non-negative values, the Author thought to have extended the original Wright function, in an original way, calling $F_{\nu}$ and $M_{\nu}$ auxiliary functions.

Presumably for this reason the function $M_{\nu}$ is referred as the Mainardi function in the book by Podlubny (Academic Press 1999) and in some papers including e.g. Balescu (CSF 2007).
It was Professor Stanković, during the presentation of the paper [Mainardi-Tomirotti (TMSF1994) at the Conference Transform Methods and Special Functions, Sofia 1994, who informed the Author that this extension for $-1 < \mu < 0$ was already made just by Wright himself in 1940 (following his previous papers in 1930’s). In his paper devoted to the 80-th birthday of Prof. Stanković, see [Mainardi-Gorenflo-Vivoli (FCAA 2005)], the Author took the occasion to renew his personal gratitude to Prof. Stanković for this earlier information that has induced him to study the original papers by Wright and work (also in collaboration) on the functions of the Wright type for further applications, see e.g. [Gorenflo-Luchko-Mainardi (1999); (2000)] and [Mainardi-Pagnini (2003)].

For more mathematical details on the functions of the Wright type, the reader may be referred to the article by [Kilbas-Saigo-Trujillo (FCAA 2002)] and references therein. For the numerical point of view we like to point out the recent paper by [Luchko (FCAA 2008)], where algorithms are provided for computation of the Wright function on the real axis with prescribed accuracy.
References


